A NOTE ON SPECIAL CURVES IN E_1^4

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Abstract

This article has consisted of a part of doctorate thesis by Süha Yılmaz [7]. Firstly, Frenet formulas are given in E_1^4. Later, characterizations of regular and inclined curves are studied in E_1^4. It has been given that a space-like curve is an inclined curve if and only if the expression

$$\frac{\kappa}{\tau} = A\cosh(\int_0^s \sigma ds) + B\sinh(\int_0^s \sigma ds),$$

obtained using harmonic curvature functions in E_1^4. In addition, it has been observed that a fifth order vectorial differential equation of position vector of a space-like curve in E_1^4 has been satisfied by means of Frenet formulas. Similarly, it has been denoted that a fourth order vectorial differential equation of tangent vector of a space-like curve in E_1^4 has been also verified using Frenet formulas. Moreover, we characterized tangent and trinormal indicatrices with one theorem. Finally, it has been denoted that if and only if spherical indicatrices of space-like curve with time-like trinormal vector are regular curves.

Keywords: Inclined curve, harmonic curvature, Frenet formulas, regular curve, space-like curve, time-like curve, spherical indicatrices.

Özet

Bu makale, Süha Yılmaz'ın doktora tezinin bir kısmını içermektedir [7]. Öncelikle, E_1^4 de Frenet formülleri verilmiştir. Daha sonra, regüler ve inclined eğrilerin karakterizasyonları E_1^4 de incelenmiştir. Ayrıca, space-like bir eğrinin inclined eğri olmasının gerek yeter şartının E_1^4 de harmonik fonksiyonlar kullanılarak elde edilen aşağıdaki koşulu sağlaması olduğu verilmiştir:

$$\frac{\kappa}{\tau} = A\cosh\left(\int_0^s \sigma ds\right) + B\sinh\left(\int_0^s \sigma ds\right)$$

İlaveten, E_1^4 deki space-like bir eğrinin konum vektörüne ait beşinci mertebeden vektörel diferensiyel denklemi, Frenet formülleri aracılığıyla sağlanmıştır. Benzer şekilde, E_1^4 deki space-like bir eğrinin teğet vektörüne ait dördüncü mertebeden vektörel diferensiyel denklemi, Frenet formülleri aracılığıyla doğrulanmıştır. Ayrıca teğet ve trinormal göstergeler bir teoremle karakterize edilmiştir. Son olarak, space-like bir eğrinin küresel göstergelerinin regüler eğri olması için gerek ve yeter koşullar verilmiştir.

Anahtar Kelimeler: Inclined eğri, harmonik eğrilik, Frenet formülleri, regüler eğri, space-like eğri, time-like eğri, küresel göstergeler.

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1. INTRODUCTION and PRELIMINARIES

The classical differential geometry of curves in Euclidean 4-space and higher dimensions is studied by Şemin [5], Gluck [3], and Mağden [8]. Also the studies of curves in Minkowski spaces are seen in the works of Yılmaz [7] and Ekmekçi [1].

In this study, we give some new characterizations of special curves in E_1^4

Let denote E_1^4 Minkowski-4 space, i.e., the manifold Euclidean 4-space E^4 together a flat with the Lorentzian metric $\langle , \rangle$ of signature (+,+,+,-) as

$$\langle X, Y \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3 - x_4 y_4,$$

where $X = (x_1, x_2, x_3, x_4)$ and $Y = (y_1, y_2, y_3, y_4)$ [4]. This metric is symmetric, bi-linear and non-degenerate one.

An arbitrary vector a in E_1^4 can have one of three Lorentzian causal characters; it can be space-like if $\langle a, a \rangle > 0$ or time-like if $\langle a, a \rangle < 0$ and null (light-like) if $\langle a, a \rangle = 0$. Similary, an arbitrary curve $\alpha = \alpha(s)$ in E_1^4 locally be space-like, time-like or null if all of its velocity vectors $\alpha'(s)$ are respectively, space-like, time-like or null for each $s \in I \subset E$. The vectors X, Y in E_1^4 are said to be orthogonal if $\langle X, Y \rangle = 0$. Recall that the norm of an arbitrary vector $a \in E_1^4$ is given by $\|a\| = \sqrt{|\langle a, a \rangle|}$ and that the velocity of the curve $\alpha(s)$ is given by $\|\alpha'(s)\|$. Therefore , $\alpha(s)$ is a unit speed curve if and only if $\langle \alpha'(s), \alpha'(s) \rangle = \pm 1$ [6].

The Lorentzian sphere of center $m = (m_1, m_2, m_3, m_4)$ and radius $r \in E^+$ in the space E_1^4 defined by $S_1^3 = \{\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in E_1^4 : \langle \alpha - m, \alpha - m \rangle = r^2\}$ [4].

Let $X = X(s)$ be a space-like or time-like curve with unit speed (arc-length parameterized) in Minkowski space time E_1^4. The Frenet frame of $X(s)$ which is shown by $\vec{T}, \vec{N}, \vec{B}$ and $\vec{E}$ can be defined as follows:

The vector $\vec{T}$ which is tangent to the curve $X(s)$ is obtained as

$$\vec{T}=\frac{d\vec{X}}{ds}, \tag{1.1}$$

and the first curvature function κ which measures the curve deviation from straight line is defined as $\kappa=\left\|\vec{T}'\right\|$.

The vector $\vec{N}$ which is called principal normal vector is defined as

$$\vec{N}=\frac{\vec{T}'}{\kappa}. \tag{1.2}$$

The third, or the binormal vector of the curve is defined as

$$\vec{B}=\frac{\vec{N}'+\kappa\vec{T}}{\left\|\vec{N}'+\kappa\vec{T}\right\|}, \tag{1.3}$$

and the second curvature function τ which measures the curve deviation from the plane $\{T,N\}$ is defined as $\tau=\left\|\vec{N}'+\kappa\vec{T}\right\|$.

The fourth, or the trinormal vector is defined as

$$\vec{E}=\mu(\vec{T}\wedge\vec{N}\wedge\vec{B}), \tag{1.4}$$

where the exterior product of $\vec{T},\vec{N},$ and $\vec{B}$ is defined as

$$\vec{T}\wedge\vec{N}\wedge\vec{B}=-\begin{vmatrix} \vec{e_1} & \vec{e_2} & \vec{e_3} & -\vec{e_4} \\ t_1 & t_2 & t_3 & t_4 \\ n_1 & n_2 & n_3 & n_4 \\ b_1 & b_2 & b_3 & b_4 \end{vmatrix}, \tag{1.5}$$

here, the vectors $\vec{e_1},\vec{e_2},\vec{e_3},$ and $\vec{e_4}$ are coordinate directions, and the frame vectors are as

$T=(t_1,t_2,t_3), N=(n_1,n_2,n_3), B=(b_1,b_2,b_3)$ in E_1^4, and also the third curvature function σ which measures the curve deviation from the subspace $\{T,N,B\}$ is defined as

$$\sigma=\mu(-\vec{B}\frac{d\vec{E}}{ds})=\mu(\vec{E}\frac{d\vec{B}}{ds}).$$

Definition 1.1. Let $X=X(s)$ be a space-like curve with unit speed, The Frenet formulas of $X=X(s)$ for any parameter $s\in I$ are defined as

$$\frac{d\vec{T}}{ds}=\kappa\vec{N},\ \frac{d\vec{N}}{ds}=-\kappa\vec{T}+\tau\vec{B},\ \frac{d\vec{B}}{ds}=-\tau\vec{N}+\sigma\vec{E},\ \frac{d\vec{E}}{ds}=\sigma\vec{B}, \tag{1.6}$$

or its matrix form is as follows

$$\begin{bmatrix}\vec{T}'\\ \vec{N}'\\ \vec{B}'\\ \vec{E}'\end{bmatrix}=\begin{bmatrix}0 & \kappa & 0 & 0\\ -\kappa & 0 & \tau & 0\\ 0 & -\tau & 0 & \sigma\\ 0 & 0 & \sigma & 0\end{bmatrix}.\begin{bmatrix}\vec{T}\\ \vec{N}\\ \vec{B}\\ \vec{E}\end{bmatrix}, \tag{1.7}$$

where $\vec{T},\vec{N},$ and $\vec{B}$ are space-like vectors and $\vec{E}$ timelike vector of the curve $X=X(s)$, and also $\kappa,\tau,$ and σ are, respectively, the first, second, and third curvature functions of the curve $X=X(s)$.

Theorem 1.1. Let $X=X(s)$ be an arbitrary parameterized space-like curve of class C^5 in E_1^4, we have

$$\vec{T}=\frac{\dot{\vec{X}}}{\left\|\dot{\vec{X}}\right\|},\ \ \vec{N}=\frac{\left\|\dot{\vec{X}}\right\|^2\ddot{\vec{X}}-(\dot{\vec{X}}.\ddot{\vec{X}})\dot{\vec{X}}}{\left\|\left\|\dot{\vec{X}}\right\|^2\ddot{\vec{X}}-(\dot{\vec{X}}.\ddot{\vec{X}})\dot{\vec{X}}\right\|},\ \ \vec{B}=\mu\vec{E}\wedge\vec{T}\wedge\vec{N},\ \text{and}\ \vec{E}=\dot{\mu}\frac{\vec{T}\wedge\vec{N}\wedge\vec{X}}{\left\|\vec{T}\wedge\vec{N}\wedge\vec{X}\right\|}, \tag{1.8}$$

$$\kappa=\frac{\left\|\left\|\dot{\vec{X}}\right\|^2\ddot{\vec{X}}-(\dot{\vec{X}}.\ddot{\vec{X}})\dot{\vec{X}}\right\|}{\left\|\dot{\vec{X}}\right\|^4},\ \ \tau=\frac{\left\|\vec{T}\wedge\vec{N}\wedge\dddot{\vec{X}}\right\|.\left\|\vec{X}\right\|}{\left\|\left\|\dot{\vec{X}}\right\|^2\ddot{\vec{X}}-(\dot{\vec{X}}.\ddot{\vec{X}})\dot{\vec{X}}\right\|},\ \ \sigma=\frac{\vec{X}^{(IV)}.\vec{E}}{\left\|\vec{T}\wedge\vec{N}\wedge\dddot{\vec{X}}\right\|}, \tag{1.9}$$

where $\cdot$ shows the derivative respect to the variable t.

Also, it is known that the curve $X=X(s)$ is a regular curve if

$$\left\|\dot{\vec{X}}\right\|\neq 0. \tag{1.10}$$

Definition 1.2. Let us consider the space-like curve $X=X(s)$. If we transport the tangent, principal normal, and the binormal vector fields to the center O of the unit hypersphere S_1^3, and the trinormal vector field to the center O of the unit hypersphere H_0^3, then we obtain spherical indicatrices of the curve $X=X(s)$ [7].

Theorem 1.2. [1] Let $\alpha=\alpha(s)$ be a regular curve with curvatures $\kappa\neq 0,\tau\neq 0,\sigma\neq 0$ in E_1^4.

The curve α is an inclined curve if and only if

$$H_1^2 - H_2^2 = const., \tag{1.11}$$

where H_1 and H_2 are the harmonic curvatures defined

$$H_1 = \frac{\kappa}{\tau} \text{ and } H_2 = \frac{H_1^{'}}{\sigma}. \tag{1.12}$$

Theorem 1.3. Let $\alpha = \alpha(s)$ be a curve in E_1^3, $\alpha = \alpha(s)$ is an inclined curve if and only if $\frac{\kappa}{\tau} = const.$ for all $s \in I$.

2. MAIN RESULTS

Theorem 2.1. Let $X = X(s)$ be space-like curve with time-like trinormal vector in L^4. The curve $X = X(s)$ in an inclined curve if and only if

$$\frac{\kappa}{\tau} = Ach\left(\int_0^s \sigma ds\right) + Bsh\left(\int_0^s \sigma ds\right),$$

where $\tau \neq 0, \sigma \neq 0; A, B$ constant.

Proof. $(\Rightarrow)$: Let the space-like curve $X = X(s)$ be helix. In this case, from Theorem 1.2,

$$H_1^2 - H_2^2 = \text{constant}. \tag{2.1}$$

Differentiating (2.1) with respect to variable s we obtain

$$H_1 H_1^{'} - H_2 H_2^{'} = 0, \tag{2.2}$$

Similarly, differentiating deriving of

$$H_2 = \frac{H_1^{'}}{\sigma}, \tag{2.3}$$

we obtain

$$H_2^{'} = \left(\frac{1}{\sigma}\right)^{'} H_1^{'} + \frac{1}{\sigma} H_1^{''}. \tag{2.4}$$

If we use (2.2), (2.3), and (2.4) we obtain

$$\sigma H_1.H_2 - H_2\left[\left(\frac{1}{\sigma}\right)^{'} H_1^{'} + \frac{1}{\sigma} H_1^{''}\right] = 0$$

or

$$H_2\left[\sigma H_1 - \left(\frac{1}{\sigma}\right)^{'} H_1^{'} + \frac{1}{\sigma} H_1^{''}\right] = 0,$$

where $H_2 \neq 0$ and $\sigma \neq 0$. Because if $\sigma = 0$ and $H_2 = 0$, then we get $H_1^{'} = 0$ and $H_1 = \frac{\kappa}{\tau} = const.$ From Theorem 1.3, this shows that the curve is an inclined curve in L^3, thus the second-order homogeneous equation with variable coefficient is obtained as follows

$$-\frac{1}{\sigma^2}.H_1'' - \frac{1}{\sigma}.\left(\frac{1}{\sigma}\right)^{'} H_1' + H_1 = 0. \tag{2.5}$$

If the transformation $t = \int_0^s \sigma ds$ is applied to H_1', H_2'', we obtain

$$H_1^{'} = \frac{dH_1}{ds} = \frac{dH_1}{dt}.\frac{dt}{ds} = \sigma H_1,$$

$$H_1^{''} = \frac{d^2H_1}{ds^2} = \frac{d^2H_1}{dt^2}.\left(\frac{dt}{ds}\right)^2 + \frac{dH_1}{dt}.\frac{d^2t}{ds^2} = \sigma^2 H_1 + \sigma^{'} H_1,$$

where indicates the derivative with respect to variable t. Substituting these equations into (2.5) we find

$-H_1 - \frac{\sigma^{'}}{\sigma^2}.H_1 + \frac{\sigma^{'}}{\sigma^2} H_1 + H_1 = 0$ therefore we obtain the following differential equation of constant coefficient

$$-\ddot{H}_1 + H_1 = 0. \tag{2.6}$$

From solution of (2.6) we get $H_1 = c_1 e^t + c_2 e^{-t}$ or $H_1 = c_1(\cosh t + \sinh t) + c_2(\cosh t - \sinh t)$,

if we say $c_1 + c_2 = A,\ c_1 - c_2 = B$ we obtain $H_1 = A\cosh t + B\sinh t$ or

$$\frac{\kappa}{\tau}=A\cosh\left(\int_0^s \sigma ds\right)+B\sinh\left(\int_0^s \sigma ds\right)$$

$(\Leftarrow)$: Let's assume that

$$\frac{\kappa}{\tau}=A\cosh\left(\int_0^s \sigma ds\right)+B\sinh\left(\int_0^s \sigma ds\right), \tag{2.7}$$

differentiating (2.7) with respect to variable s we get

$$\frac{1}{\sigma}\frac{d}{ds}\frac{\kappa}{\tau}=A\cosh\left(\int_0^s \sigma ds\right)+B\sinh\left(\int_0^s \sigma ds\right), \tag{2.8}$$

similarly, differentiating (2.8) with respect to variable s gives

$\frac{d}{ds}\left[\frac{1}{\sigma}\frac{d}{ds}\frac{\kappa}{\tau}\right]=A\cosh\left(\int_0^s \sigma ds\right)+B\sinh\left(\int_0^s \sigma ds\right)$ or $H_2^{'}=\sigma H_1$ and here we obtain

$$H_2^{'}-\sigma H_1=0, \tag{2.9}$$

multiplying both side of the expression (2.9) with $H_2=\frac{1}{\sigma}H_1^{'}$, we obtain

$$H_1H_1^{'}-H_2H_2^{'}=0, \tag{2.10}$$

taking the integral of both sides of (2.10), we get

$H_1^2-H_2^2=$const., which indicates that the curve is an inclined curve in L^4 from Theorem 1.2.

Theorem 2.2. Let $X : I \to L^4$ be an arc length parametrized space-like curve from class C^4 such that $\chi > 0, \sigma > 0$ and $\sigma = const$. Then position vector $X = X(s)$ of the curve satisfies following a vectoral differential equation of 5^{th} order

$$\frac{\kappa^3}{\tau}.\vec{X}^{(v)} + \left[\left(\frac{\kappa^3}{\tau}\right)' + \left(\frac{1}{\tau}\right).\frac{1}{\kappa} - \frac{2\kappa'}{\tau\kappa^2} - \frac{\sigma^2}{\kappa^\tau}\right]\vec{X}^{(\imath v)}$$

$$+2\left[\left(\frac{1}{\tau}\right)''.\frac{1}{\kappa} + \left(\frac{1}{\tau}\right)'.\left(\frac{1}{\kappa}\right)' - 2\left(\frac{\kappa'}{\tau\kappa^2}\right)' - \left(\frac{\sigma^2}{\kappa\tau}\right)' - \frac{\kappa''}{\kappa^2}\left(\frac{1}{\tau}\right)' - \frac{\kappa''\kappa^2}{\tau} + 2.\frac{\kappa'^2}{\kappa^3\tau} + \frac{\kappa}{\tau} + \frac{\tau}{\kappa} - \frac{\sigma^2\kappa'}{\kappa^2\tau}\right]\vec{X}'''$$

$$+\left[-\frac{\kappa'}{\kappa^2}\left(\frac{1}{\tau}\right)' - \frac{\kappa'}{\kappa^2}\left(\frac{1}{\tau}\right)'' + \left(\frac{\kappa''\kappa^2}{\tau}\right) + 2\left(\frac{\kappa'^2}{\kappa^3\tau}\right) + \left(\frac{\kappa}{\tau}\right)' + \left(\frac{\tau}{\kappa}\right)' - \left(\frac{\sigma^2\kappa'}{\tau\kappa^2}\right)' + \kappa\left(\frac{1}{\tau}\right)' + \frac{\kappa'}{\tau} - \frac{\sigma^2}{\tau}\kappa\right]\vec{X}''$$

$$+\left[\kappa'\left(\frac{1}{\tau}\right)' + \kappa\left(\frac{1}{\tau}\right)'' + \left(\frac{\kappa'}{\tau}\right)' - \left(\frac{\sigma^2\kappa}{\tau}\right)'\right]\vec{X}' = 0,$$

where $\vec{T}, \vec{N}, \vec{B}$ space-like vectors, $\vec{E}$ time-like vector, σ constant.

Proof. Suppose $\vec{T}, \vec{N}, \vec{B}$ are space-like vectors, $\vec{E}$ is time-like vector for space-like curve $X = X(s)$ in L^4 from $(1.6)_1$ and $(1.6)_2$ we obtain

$$\vec{N} = \frac{\vec{T}'}{\kappa} \tag{2.11}$$

$$\vec{B} = \frac{1}{\tau}(\kappa\vec{T} + \vec{N}') \tag{2.12}$$

Substituting (2.11) in $(1.6)_3$ we get

$$\vec{B}' = -\frac{\tau}{\kappa}\vec{T}' + \sigma\vec{E} \tag{2.13}$$

differentiating (2.11) and substituting it into (2.12) we find

$$\vec{B} = \frac{1}{\tau}\left(\kappa\vec{T} + \left(\frac{\vec{T}'}{\kappa}\right)'\right) \tag{2.14}$$

taking the integral of both sides of $(1.6)_4$ we get

$$\vec{E} = \int \sigma \vec{B} ds \tag{2.15}$$

and substituting (2.14) into (2.15) we obtain

$$\vec{E} = \int \frac{\sigma}{\tau}\left[\kappa \vec{T} + \left(\frac{\vec{T}'}{\kappa}\right)'\right] ds \tag{2.16}$$

and substituting (2.16) into (2.13) we find

$$\vec{B}' = -\frac{\tau}{\kappa}\vec{T}' + \sigma^2 \int \frac{1}{\tau}\left[\kappa \vec{T} + \left(\frac{\vec{T}'}{\kappa}\right)'\right] ds \tag{2.17}$$

similarly differentiating (2.14) and using (2.17) we obtain

$$\begin{aligned} &\left(\frac{1}{\tau}\right)'\left[\frac{\vec{T}''\kappa - \vec{T}'\kappa'}{\kappa^2} + \kappa \vec{T}\right] + \frac{1}{\tau}\left[\left(\vec{T}'''\kappa - \vec{T}'\kappa''\right)\kappa^2 - \frac{2\kappa\kappa'\left(\vec{T}''\kappa - \vec{T}'\kappa'\right)}{\kappa^4} + \kappa'\vec{T} + \kappa\vec{T}'\right] \\ &+ \frac{\tau}{\kappa}\vec{T}' - \sigma^2 \int \frac{1}{\tau}\left[\kappa \vec{T} + \left(\frac{\vec{T}'}{\kappa}\right)'\right] ds = 0. \end{aligned} \tag{2.18}$$

Differentiating (2.18) and substituting

$\dot{\vec{X}} = \vec{T}, \ddot{\vec{X}} = \vec{T}', \dddot{\vec{X}} = \vec{T}'', \vec{X}^{(IV)} = \vec{T}''', \vec{X}^{(V)} = \vec{T}^{(IV)}$ into this expression we obtain

$$\begin{aligned} &\frac{\kappa^3}{\tau}\vec{X}^{(v)} + \left[\left(\frac{\kappa^3}{\tau}\right)' + \left(\frac{1}{\tau}\right).\frac{1}{\kappa} - \frac{2\kappa'}{\tau\kappa^2} - \frac{\sigma^2}{\kappa\tau}\right]\vec{X}^{(iv)} \\ &+ \left[\left(\frac{1}{\tau}\right)''.\frac{1}{\kappa} + \left(\frac{1}{\tau}\right)'\left(\frac{1}{\kappa}\right)' - 2\left(\frac{\kappa'}{\tau\kappa^2}\right)' - \left(\frac{\sigma^2}{\kappa\tau}\right)' - \frac{\kappa''}{\kappa^2}\left(\frac{1}{\tau}\right)' - \frac{\kappa''\kappa^2}{\tau} + 2.\frac{\kappa'^2}{\kappa^3\tau} + \frac{\kappa}{\tau} + \frac{\tau}{\kappa} - \frac{\sigma^2\kappa'}{\kappa^2\tau}\right]\vec{X}''' \\ &+ \left[-\frac{\kappa'}{\kappa^2}\left(\frac{1}{\tau}\right)' - \frac{\kappa'}{\kappa^2}\left(\frac{1}{\tau}\right)'' + \left(\frac{\kappa''\kappa^2}{\tau}\right)' + 2\left(\frac{\kappa'^2}{\kappa^3\tau}\right) + \left(\frac{\kappa}{\tau}\right)' + \left(\frac{\tau}{\kappa}\right)' - \left(\frac{\sigma^2\kappa'}{\tau\kappa^2}\right)' + \kappa\left(\frac{1}{\tau}\right)' + \frac{\kappa'}{\tau} - \frac{\sigma^2}{\tau}\kappa\right]\vec{X}'' \\ &+ \left[\kappa'\left(\frac{1}{\tau}\right)' + \kappa\left(\frac{1}{\tau}\right)'' + \left(\frac{\kappa'}{\tau}\right)' - \left(\frac{\sigma^2\kappa}{\tau}\right)'\right]\vec{X}' = 0. \end{aligned}$$

Denoting these coefficients of the vectoral differential equation by

$$P(s), Q(s), R(s), K(s), L(s).$$

Respectively, then we can rewrite this last equation as

$$P(s)\vec{X}^{(v)} + Q(s)\vec{X}^{(iv)} + R(s)\vec{X}''' + K(s)\vec{X}'' + L(s)\vec{X}' = 0, \tag{2.19}$$

where

$$P(s) = \frac{\kappa^3}{\tau},$$

$$Q(s) = \left(\frac{\kappa^3}{\tau}\right)' + \left(\frac{1}{\tau}\right)'.\frac{1}{\kappa} - \frac{2\kappa'}{\tau\kappa^2} - \frac{\sigma^2}{\kappa\tau},$$

$$R(s) = \left[\left(\frac{1}{\tau}\right)''.\frac{1}{\kappa} + \left(\frac{1}{\tau}\right)'\left(\frac{1}{\kappa}\right)' - 2\left(\frac{\kappa'}{\tau\kappa^2}\right)' - \left(\frac{\sigma^2}{\kappa\tau}\right)' - \frac{\kappa''}{\kappa^2}\left(\frac{1}{\tau}\right)' - \frac{\kappa''\kappa^2}{\tau} + 2\frac{\kappa'^2}{\kappa^3\tau} + \frac{\kappa}{\tau} + \frac{\tau}{\kappa} - \frac{\sigma^2\kappa'}{\kappa^2\tau}\right],$$

$$K(s) = \left[-\frac{\kappa'}{\kappa^2}\left(\frac{1}{\tau}\right)' - \frac{\kappa'}{\kappa^2}\left(\frac{1}{\tau}\right)'' + \left(\frac{\kappa''\kappa^2}{\tau}\right)' + 2\left(\frac{\kappa'^2}{\kappa^3\tau}\right) + \left(\frac{\kappa}{\tau}\right)' + \left(\frac{\tau}{\kappa}\right)' - \left(\frac{\sigma^2\kappa'}{\tau\kappa^2}\right)' + \kappa\left(\frac{1}{\tau}\right)' + \frac{\kappa'}{\tau} - \frac{\sigma^2}{\tau}\kappa\right],$$

$$L(s) = \kappa'\left(\frac{1}{\tau}\right)' + \kappa\left(\frac{1}{\tau}\right)'' + \left(\frac{\kappa'}{\tau}\right)' - \left(\frac{\sigma^2\kappa}{\tau}\right)',$$

κ is the first curvature, τ the second curvature, σ the third curvature and constant.

The equation (2.19) has a solution by Chebyshev-Matrix Method [2]. If , $P(s), Q(s), R(s), K(s)$ and $L(s)$ are one-variable functions, then these functions satisfy the following conditions:

i) They must be satisfied by 5th order differential equation .
ii) They must be formed into Taylor series.

If $\kappa \neq 0, \tau \neq 0$ in ordinary differential equation (2.19), they can be formed Maclaurin series.

In this situation, the equation (2.19) is transformed to a matrix form with Chebyshev-Matrix Method. Thus, solution of the equation (2.19) can be obtained with an analytic or approximate method.

Theorem 2.3. Let $X = X(s)$ be arc length parametrized space-like curve with time-like trinormal vector from class C^4 in L^4 such that $\kappa > 0, \tau > 0, \sigma > 0$. Then tangent vector $\vec{T}$ of $X = X(s)$ satisfies following a vectoral differential equation of 4^{th} order

$$\vec{T}^{(iv)} + \left[\left(\frac{\kappa}{\tau}\right)^{''}\frac{\tau}{\kappa} + 2\left[\ln\left|\frac{\kappa}{\tau}\right|\right]^{'} + \left(\frac{\sigma}{\kappa}\right)^{'}.\frac{1}{\sigma}\right]\vec{T}^{'''} + \left[\left[\ln\left|\frac{\kappa}{\tau}\right|\right]^{'}.\left(\frac{\sigma}{\kappa}\right)^{'}.\frac{1}{\sigma} - \frac{\sigma^2}{\kappa} - 1\right]\vec{T}^{''}$$
$$+\left[\left(\frac{\sigma}{\kappa}\right)^{'}.\frac{\tau^2}{\kappa\sigma} - \left(\frac{\tau}{\kappa}\right)\left(\frac{\tau}{\kappa}\right)^{'} - \left(\frac{\sigma}{\kappa}\right)^{'}.\frac{1}{\sigma} - 2\left[\ln\left|\frac{\kappa}{\tau}\right|\right]^{'}\right]\vec{T}^{'} + \left[-\left(\frac{\kappa}{\tau}\right)^{''}.\left(\frac{\tau}{\kappa}\right) - \left[\ln\left|\frac{\kappa}{\tau}\right|\right]^{'}.\left(\frac{\sigma}{\kappa}\right)^{'}.\frac{1}{\sigma} - \frac{\sigma^2}{\kappa}\right]\vec{T} = 0$$

Proof. Suppose $\vec{T}, \vec{N}, \vec{B}$ are space-like vectors and $\vec{E}$ is a time-like vector, taking the derivative of Frenet formulas (1.6) with respect to arc length parametrized s_t of tangent indicatrix, we obtain

$$\begin{aligned} \vec{T}^{'} &= \vec{N}, \\ \vec{N}^{'} &= -\vec{T} + \frac{\tau}{\kappa}\vec{B}, \\ \vec{B}^{'} &= -\frac{\tau}{\kappa}\vec{N} + \frac{\sigma}{\kappa}\vec{E}, \\ \vec{E}^{'} &= \frac{\sigma}{\kappa}\vec{B}, \end{aligned} \qquad (2.20)$$

From $(2.20)_2$ and $(2.20)_3$ we get

$$\vec{B} = \frac{\kappa}{\tau}(\vec{N}^{'} - \vec{T}), \qquad (2.21)$$

$$\vec{E} = \frac{\tau}{\sigma}\vec{N} + \frac{\kappa}{\sigma}\vec{B}^{'}, \qquad (2.22)$$

And substituting (2.21) into $(2.20)_4$ we get

$$\vec{E}^{'} = \frac{\sigma}{\tau}(\vec{T} - \vec{N}^{'}) \qquad (2.23)$$

Differentiating (2.21) and using $(2.20)_3$ we have

$$\left(\frac{\kappa}{\tau}\right)'(\vec{T} - \vec{N}') + \left(\frac{\kappa}{\tau}\right)(\vec{T}' - \vec{N}'') = \frac{\tau}{\kappa}\vec{N} - \frac{\sigma}{\kappa}\vec{E} \qquad (2.24)$$

taking the first derivative of (2.21) and substituting $(2.20)_1$, (2.21), (2.22) and (2.23) in this expression we obtain,

$$\vec{T}^{(iv)}+\left[\left(\frac{\kappa}{\sigma}\right)^{''}\frac{\tau}{\chi}+2\left[\ln\left|\frac{\kappa}{\tau}\right|\right]^{'}+\left(\frac{\sigma}{\kappa}\right)^{'}.\frac{1}{\sigma}\right]\vec{T}^{'''}+\left[\left[\ln\left|\frac{\kappa}{\tau}\right|\right]^{'}.\left(\frac{\sigma}{\kappa}\right)^{'}.\frac{1}{\sigma}-\frac{\sigma^2}{\kappa}-1\right]\vec{T}^{''}$$
$$+\left[\left(\frac{\sigma}{\kappa}\right)^{'}.\frac{\tau^2}{\kappa\sigma}-\left(\frac{\tau}{\kappa}\right).\left(\frac{\tau}{\kappa}\right)^{'}-\left(\frac{\sigma}{\kappa}\right)^{'}\frac{1}{\sigma}-2\left[\ln\left|\frac{\kappa}{\tau}\right|\right]^{'}\right]\vec{T}^{'} \tag{2.25}$$
$$+\left[-\left(\frac{\kappa}{\tau}\right)^{''}.\left(\frac{\tau}{\kappa}\right)-\left[\ln\left|\frac{\kappa}{\tau}\right|\right]^{'}.\left(\frac{\sigma}{\kappa}\right)^{'}.\frac{1}{\sigma}-\frac{\sigma^2}{\kappa}\right]\vec{T}=0$$

Theorem 2.4. Suppose $X=X(s), Y=Y(s)$ are space-like curves with time-like trinormal vector in L^4 and let the first curvature of $X=X(s)$ be constant. If trinormal indicatrix of $X=X(s)$ is tangent indicatrix of $Y=Y(s)$, then the third curvature of $Y=Y(s)$ is constant.

Proof. Let's calculate Frenet formulas of curve $Y=Y(s)$. Let $\vec{T}_x,\vec{N}_x,\vec{B}_x,\vec{E}_x,\kappa_x,\tau_x,\sigma_x$, and

$\vec{T}_y,\vec{N}_y,\vec{B}_y,\vec{E}_y,\kappa_y,\tau_y,\sigma_y$ be Frenet elements of curve $X=X(s)$ and $Y=Y(s)$, respectively. Suppose s_y be arc length parametrized of $Y=Y(s)$. Then, we can write

$$\vec{Y}=\int\vec{E}_x(s)ds, \tag{2.26}$$

differentiating both sides of (2.26) with respect to s, we have

$$\frac{d\vec{Y}}{ds}=\frac{dY}{ds_y}.\frac{ds_y}{ds}=\vec{E}_x\,.$$

Since

$$\frac{d\vec{Y}}{ds_y}=\vec{T}_y\,, \tag{2.27}$$

we get

$$\vec{T}_y.\frac{ds_y}{ds}=\vec{E}_x\,, \tag{2.28}$$

and using this expression, we obtain

$$\vec{T}_y = \vec{E}_x \text{ and } \frac{ds_y}{ds} = 1, \tag{2.29}$$

taking the derivatives of both side of (2.29) with respect to s, we get $\chi_y.\vec{N}_y = \sigma_x.\vec{B}_x$ and here we obtain $\vec{N}_y = \vec{B}_x$ and $\chi_y = \sigma_x$, taking the derivative of both sides of expression $\vec{N}_y = \vec{B}_x$ with respect to s we obtain

$$-\kappa_y.\vec{T}_y + \tau_y.\vec{B}_y = -\tau_x.\vec{N}_x + \sigma_x.\vec{E}_x$$

and the expression, we have

$$\tau_y = \tau_x \text{ and } \vec{B}_y = -\vec{N}_x. \tag{2.30}$$

If exterior product of $\vec{T}_y \wedge \vec{N}_y \wedge \vec{B}_y$ is formed, we obtain

$$\vec{E}_y = -\vec{T}_x, \tag{2.31}$$

and taking the derivative of both side of (2.30) with respect to s, we get

$$\sigma_y \vec{B}_y = -\kappa_x \vec{N}_x. \tag{2.32}$$

Since $\vec{B}_y = -\vec{N}_x$ from $(2.30)_2$ we get

$$\sigma_y = \kappa_x. \tag{2.33}$$

ince κ_x is constant, σ_y is also found as constant by using (2.33) .

Theorem 2.5. Let tangents, principal normals, binormals and trinormals indicatrices of space-like curve X be X_1, X_2, X_3, X_4, respectively. If spherical indicatrices of

$$X : I \to L^4$$

$$s \to X(s)$$

satisfy following conditions, then they are regular curve

i) The curve X_1 is regular $\Leftrightarrow \kappa \neq 0$

ii) The curve X_2 is regular $\Leftrightarrow \sqrt{\kappa^2 - \tau^2} \neq 0 \;\; (\kappa \neq 0, \tau \neq 0)$

iii) The curve X_3 is regular $\Leftrightarrow \sqrt{\left|\tau^2 - \sigma^2\right|} \neq 0 \;\; (\tau \neq 0, \sigma \neq 0)$

iv) The curve X_4 is regular $\Leftrightarrow \sigma \neq 0$

Proof. From (1.10)

i) Since,

$$X_1 = \vec{T} \Rightarrow \frac{dX_1}{ds} = \kappa \vec{N},$$

the curve X_1 is regular $\Leftrightarrow \left\|\frac{dX_1}{ds}\right\| = \kappa \neq 0$

ii) Similarly, since

$$X_2 = \vec{N} \Rightarrow \frac{dX_2}{ds} = -\kappa \vec{T} + \tau \vec{B},$$

the curve X_2 is regular $\Leftrightarrow \left\|\frac{dX_2}{ds}\right\| = \sqrt{\kappa^2 + \tau^2} \;\;\; (\kappa \neq 0, \tau \neq 0)$

iii) Also, since

$$X_3 = \vec{B} \Rightarrow \frac{dX_3}{ds} = -\tau \vec{N} + \sigma \vec{E},$$

the curve X_3 is regular $\Leftrightarrow \frac{dX_3}{ds} = \sqrt{\left|\tau^2 - \sigma^2\right|} \;\; (\tau \neq 0, \sigma \neq 0)$

iv) Finally, since

$$X_4 = \vec{E} \Rightarrow \frac{dX_4}{ds} = \sigma \vec{B},$$

The curve X_4 is regular $\Leftrightarrow \left\| \frac{dX_2}{ds} \right\| = \sigma \neq 0$. with time-like trinormal vector.

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