Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(2): 1505–1518.
DOI:10.3934/math.2020103
Received: 05 November 2019
Accepted: 09 January 2020
Published: 21 January 2020

Research article

Some integral inequalities of Hermite-Hadamard type for multiplicatively preinvex functions

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Abstract: In this paper, some integral inequalities of Hermite-Hadamard type for multiplicatively preinvex functions are established. Also, new inequalities involving multiplicative integrals are obtained for product and quotient of preinvex and multiplicatively preinvex functions.

Keywords: invex sets; preinvex functions; multiplicative calculus; Hermite-Hadamard inequalities
Mathematics Subject Classification: 26D07, 26D15

1. Introduction

The function $f : I \subset \mathbb{R} \to \mathbb{R}$, is said to be convex, if the inequality

$$f(t\alpha + (1-t)\beta) \leq tf(\alpha) + (1-t) f(\beta)$$

holds for all $\alpha, \beta \in I$ and $t \in [0, 1]$. We say that f is concave if $(-f)$ is convex.
The concept of convexity is one of the most important research area in many branches of pure and applied mathematics. It has a key role in many fields of applications, especially in optimization theory and the theory of inequalities. A useful inequality for convex functions is given as follows:
Let f is a convex function on the interval $I = [\alpha, \beta]$ of real numbers with $\alpha < \beta$, then

$$f\left(\frac{\alpha+\beta}{2}\right) \leq \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} f(x)\,dx \leq \frac{f(\alpha)+f(\beta)}{2}, \quad \alpha, \beta \in I. \tag{1.1}$$

This double inequality is well known in the literature as Hermite-Hadamard integral inequality for convex functions (see, e.g., [7, 21]). Both inequalities hold in the reversed direction if f is concave. Recently, Hermite-Hadamard integral inequality for convex functions has received renewed attention by many researchers and as gradually a remarkable of generalizations, extensions and refinements in various directions have been found, see [1, 8, 10, 13, 19, 20, 23, 24, 26] and the references cited therein.

In recent years, several generalizations and extensions have been considered for the concept of the convexity. One of the most important generalizations of convex functions is that of invex functions introduced by Hanson [9]. Ben-Israel and Mond [6] introduced the notions of invex sets and preinvex functions. Weir and Mond [25] and Noor [18] have studied the basic properties of the preinvex functions. For recent applications and generalizations of the preinvex functions, we refer to [3, 4, 11, 12, 14, 16, 17].

2. Preliminaries

2.1. Preinvexity and Hermite-Hadamard inequality

Let $\Im$ be a nonempty closed set in $\mathbb{R}^n$. Let $f : \Im \to \mathbb{R}$ be a continuous function and let $\eta(.,.) : \Im \times \Im \to \mathbb{R}$ be a continuous bifunction. Now, we recall some well known concepts.

Definition 2.1. *[27] The set $\Im$ is said to be invex with respect to $\eta(.,.)$, if*

$$\alpha + t\eta(\beta, \alpha) \in \Im, \ \ \forall \alpha, \beta \in \Im, \ \ t \in [0, 1].$$

The invex set $\Im$ is also called a η-connected set.

It is true that every convex set is an invex set with respect to $\eta(\beta, \alpha) = \beta - \alpha$, but the converse is not necessarily true, see [25, 28] and the references therein. For the sake of simplicity, we always assume that $\Im = [\alpha, \alpha + \eta(\beta, \alpha)]$, unless otherwise specified [2].

Definition 2.2. *[25] Let f be a function on the invex set $\Im$. Then, f is said to be preinvex with respect to η, if*

$$f(\alpha + t\eta(\beta, \alpha)) \leq (1 - t) f(\alpha) + t f(\beta), \ \ \forall \alpha, \beta \in \Im, \ \ t \in [0, 1].$$

The function f is said to be preconcave if and only if $-f$ is preinvex.

Note that every convex function is a preinvex function, but the converse is not true. For example, the function $f(\alpha) = -|\alpha|$ is not a convex function, but it is a preinvex function with respect to η, where

$$\eta(\beta, \alpha) = \begin{cases} \alpha - \beta, & if \ \ \beta \leq 0, \ \alpha \leq 0 \ \ and \ \ \beta \geq 0, \ \alpha \geq 0 \\ \beta - \alpha, & otherwise \end{cases}$$

Definition 2.3. *[22] Let f be a function on the invex set $\Im$. Then, f is said to be prequasiinvex with respect to η, if*

$$f(\alpha + t\eta(\beta, \alpha)) \leq \max\{f(\alpha), f(\beta)\}, \ \ \forall \alpha, \beta \in \Im, \ \ t \in [0, 1].$$

Definition 2.4. *[18] The function f on the invex set $\Im$ is said to be multiplicatively preinvex with respect to η, if*

$$f(\alpha + t\eta(\beta, \alpha)) \leq (f(\alpha))^{1-t} (f(\beta))^t, \ \ \forall \alpha, \beta \in \Im, \ \ t \in [0, 1].$$

From the above definitions, we have

$$\begin{aligned} f(\alpha + t\eta(\beta, \alpha)) &\leq (f(\alpha))^{1-t} (f(\beta))^t \\ &\leq (1 - t) f(\alpha) + t f(\beta) \end{aligned}$$

$$\leq \max\{f(\alpha), f(\beta)\}.$$

To prove some results in the paper, we need the well-known Condition C introduced by Mohan and Neogy in [15].

Condition C. Let $\mathfrak{I} \subseteq \mathbb{R}^n$ be an open invex subset with respect to $\eta : \mathfrak{I} \times \mathfrak{I} \to \mathbb{R}$. We say that the bifunction η satisfies the Condition C if for any $\alpha, \beta \in \mathfrak{I}$ and $t \in [0, 1]$,

$$\eta(\alpha, \alpha + t\eta(\beta, \alpha)) = -t\eta(\beta, \alpha),$$

$$\eta(\beta, \alpha + t\eta(\beta, \alpha)) = (1 - t)\eta(\beta, \alpha).$$

Note that for every $\alpha, \beta \in \mathfrak{I}$ and $t \in [0, 1]$ and from condition C, we have

$$\eta(\alpha + t_2\eta(\beta, \alpha), \alpha + t_1\eta(\beta, \alpha)) = (t_2 - t_1)\eta(\beta, \alpha).$$

In [16] Noor has obtained the following Hermite-Hadamard inequalities for the preinvex functions.

Theorem 2.5. *Let $f : \mathfrak{I} = [\alpha, \alpha + \eta(\beta, \alpha)] \to (0, \infty)$ be a preinvex function on the interval of real numbers $\mathfrak{I}^\circ$ and $\alpha, \beta \in \mathfrak{I}^\circ$ with $\alpha < \alpha + \eta(\beta, \alpha)$. Then the following inequality holds:*

$$f\left(\frac{2\alpha + \eta(\beta, \alpha)}{2}\right) \leq \frac{1}{\eta(\beta, \alpha)} \int_\alpha^{\alpha+\eta(\beta,\alpha)} (f(x))\, dx \leq \frac{f(\alpha) + f(\beta)}{2}.$$

2.2. Multiplicative calculus

Recall that the concept of multiplicative integral called * integral is denoted by $\int_a^b (f(x))^{dx}$ while the ordinary integral is denoted by $\int_a^b (f(x))\, dx$. This comes from the fact that the sum of the terms of product is used in the definition of a classical Riemann integral of f on $[a, b]$, the product of terms raised to certain powers is used in the definition of multiplicative integral of f on $[a, b]$.

There is the following relation between Riemann integral and * integral [5].

Proposition 2.6. *If f is Riemann integrable on $[a, b]$, then f is * integrable on $[a, b]$ and*

$$\int_a^b (f(x))^{dx} = e^{\int_a^b \ln(f(x))dx}.$$

In [5], Bashirov et al. show that * integral has the following results and notations:

Proposition 2.7. *If f is positive and Riemann integrable on $[a, b]$, then f is * integrable on $[a, b]$ and*

1. $\int_a^b ((f(x))^p)^{dx} = \int_a^b \left((f(x))^{dx}\right)^p$,
2. $\int_a^b (f(x)g(x))^{dx} = \int_a^b (f(x))^{dx} \cdot \int_a^b (g(x))^{dx}$,
3. $\int_a^b \left(\frac{f(x)}{g(x)}\right)^{dx} = \frac{\int_a^b (f(x))^{dx}}{\int_a^b (g(x))^{dx}}$,
4. $\int_a^b (f(x))^{dx} = \int_a^c (f(x))^{dx} \cdot \int_c^b (f(x))^{dx}$, $a \leq c \leq b$.
5. $\int_a^a (f(x))^{dx} = 1$ and $\int_a^b (f(x))^{dx} = \left(\int_b^a (f(x))^{dx}\right)^{-1}$.

3. Main results

In this section we establish some integral inequalities of Hermite-Hadamard type for multiplicatively preinvex positive functions. We also obtain new integral inequalities for product and quotient of preinvex and multiplicatively preinvex positive functions in the framework of multiplicative calculus.

Theorem 3.1. *Let $\Im \subseteq \mathbb{R}$ an open invex subset with respect to $\eta : \Im \times \Im \to \mathbb{R}$ and $u_1, u_2 \in \Im$ with $u_1 < u_1 + \eta(u_2, u_1)$. If f is a positive and multiplicatively preinvex function on the interval $[u_1, u_1 + \eta(u_2, u_1)]$ and η satisfies Condition C, then*

$$f\left(\frac{2u_1 + \eta(u_2, u_1)}{2}\right) \le \left(\int_{u_1}^{u_1+\eta(u_2,u_1)} (f(x))^{dx}\right)^{\frac{1}{\eta(u_2,u_1)}} \le G(f(u_1), f(u_2)), \tag{3.1}$$

where $G(.,.)$ is a geometric mean.

(3.1) is called Hermite-Hadamard integral inequalities for multiplicatively preinvex functions.

Proof. Since f is a multiplicatively preinvex function, we have for every $\alpha, \beta \in [u_1, u_1 + \eta(u_2, u_1)]$ with $t = \frac{1}{2}$

$$f\left(\frac{2\alpha + \eta(\beta, \alpha)}{2}\right) = f\left(\alpha + \frac{\eta(\beta, \alpha)}{2}\right) \le (f(\alpha))^{\frac{1}{2}} (f(\beta))^{\frac{1}{2}}.$$

Now, let $\alpha = u_1 + (1-t)\eta(u_2, u_1)$ and $\beta = u_1 + t\eta(u_2, u_1)$. From Condition C, we have

$$\begin{aligned} & f\left(u_1 + (1-t)\eta(u_2, u_1) + \frac{\eta(u_1 + t\eta(u_2, u_1), u_1 + (1-t)\eta(u_2, u_1))}{2}\right) \\ = \; & f\left(u_1 + (1-t)\eta(u_2, u_1) + \frac{(2t-1)\eta(u_2, u_1)}{2}\right) \\ = \; & f\left(\frac{2u_1 + \eta(u_2, u_1)}{2}\right) \\ \le \; & (f(u_1 + t\eta(u_2, u_1)))^{\frac{1}{2}} (f(u_1 + (1-t)\eta(u_2, u_1)))^{\frac{1}{2}}. \end{aligned}$$

Taking logarithms of both sides of the above inequality leads to

$$\begin{aligned} \ln f\left(\frac{2u_1 + \eta(u_2, u_1)}{2}\right) &\le \ln\left((f(u_1 + t\eta(u_2, u_1)))^{\frac{1}{2}} . (f(u_1 + (1-t)\eta(u_2, u_1)))^{\frac{1}{2}}\right) \\ &= \frac{1}{2}\ln(f(u_1 + t\eta(u_2, u_1))) + \frac{1}{2}\ln(f(u_1 + (1-t)\eta(u_2, u_1))). \end{aligned}$$

Integrating the above inequality with respect to t on $[0, 1]$, we have

$$\begin{aligned} \ln f\left(\frac{2u_1 + \eta(u_2, u_1)}{2}\right) &\le \frac{1}{2}\int_0^1 \ln(f(u_1 + t\eta(u_2, u_1)))\,dt + \frac{1}{2}\int_0^1 \ln(f(u_1 + (1-t)\eta(u_2, u_1)))\,dt \\ &= \frac{1}{2}\left[\frac{1}{\eta(u_2, u_1)}\int_{u_1}^{u_1+\eta(u_2,u_1)} \ln(f(x))\,dx - \frac{1}{\eta(u_2, u_1)}\int_{u_1+\eta(u_2,u_1)}^{u_1} \ln(f(x))\,dx\right] \end{aligned}$$

$$= \frac{1}{2}\left[\frac{1}{\eta(u_2,u_1)}\int_{u_1}^{u_1+\eta(u_2,u_1)}\ln(f(x))\,dx + \frac{1}{\eta(u_2,u_1)}\int_{u_1}^{u_1+\eta(u_2,u_1)}\ln(f(x))\,dx\right]$$
$$= \frac{1}{\eta(u_2,u_1)}\int_{u_1}^{u_1+\eta(u_2,u_1)}\ln(f(x))\,dx.$$

Thus,

$$\begin{aligned} f\left(\frac{2u_1+\eta(u_2,u_1)}{2}\right) &\le e^{\left(\frac{1}{\eta(u_2,u_1)}\int_{u_1}^{u_1+\eta(u_2,u_1)}\ln(f(x))dx\right)} \\ &= \left(\int_{u_1}^{u_1+\eta(u_2,u_1)}(f(x))^{dx}\right)^{\frac{1}{\eta(u_2,u_1)}}. \end{aligned}$$

Hence, we have

$$f\left(\frac{2u_1+\eta(u_2,u_1)}{2}\right) \le \left(\int_{u_1}^{u_1+\eta(u_2,u_1)}(f(x))^{dx}\right)^{\frac{1}{\eta(u_2,u_1)}}, \tag{3.2}$$

which completes the proof of the first inequality in (3.1).
Now consider the second inequality in (3.1).

$$\begin{aligned} \left(\int_{u_1}^{u_1+\eta(u_2,u_1)}(f(x))^{dx}\right)^{\frac{1}{\eta(u_2,u_1)}} &= \left(e^{\left(\int_{u_1}^{u_1+\eta(u_2,u_1)}\ln(f(x))dx\right)}\right)^{\frac{1}{\eta(u_2,u_1)}} \\ &= e^{\frac{1}{\eta(u_2,u_1)}\left(\int_{u_1}^{u_1+\eta(u_2,u_1)}\ln(f(x))dx\right)} \\ &= e^{\left(\int_0^1 \ln(f(u_1+t\eta(u_2,u_1)))dt\right)} \\ &\le e^{\left(\int_0^1 \ln\left((f(u_1))^{1-t}(f(u_2))^t\right)dt\right)} \\ &= e^{\left(\int_0^1 ((1-t)\ln f(u_1)+t\ln f(u_2))dt\right)} \\ &= e^{\left(\ln(f(u_1).f(u_2))^{\frac{1}{2}}\right)} \\ &= \sqrt{f(u_1).f(u_2)} \\ &= G(f(u_1),f(u_2)). \end{aligned}$$

Hence, we get the inequality

$$\left(\int_{u_1}^{u_1+\eta(u_2,u_1)}(f(x))^{dx}\right)^{\frac{1}{\eta(u_2,u_1)}} \le G(f(u_1),f(u_2)). \tag{3.3}$$

Combining the inequalities (3.2) and (3.3), we have

$$f\left(\frac{2u_1+\eta(u_2,u_1)}{2}\right) \le \left(\int_{u_1}^{u_1+\eta(u_2,u_1)}(f(x))^{dx}\right)^{\frac{1}{\eta(u_2,u_1)}} \le G(f(u_1),f(u_2)),$$

which is the desired result. □

Theorem 3.2. *Let $\Im \subseteq \mathbb{R}$ an open invex subset with respect to $\eta : \Im \times \Im \to \mathbb{R}$ and $u_1, u_2 \in \Im$ with $u_1 < u_1 + \eta(u_2, u_1)$. If f and g are positive and multiplicatively preinvex functions on the interval $[u_1, u_1 + \eta(u_2, u_1)]$ and η satisfies Condition C, then*

$$\begin{aligned} & f\left(\frac{2u_1 + \eta(u_2, u_1)}{2}\right) g\left(\frac{2u_1 + \eta(u_2, u_1)}{2}\right) \\ \leq & \left(\int_{u_1}^{u_1+\eta(u_2,u_1)} (f(x))^{dx} \cdot \int_{u_1}^{u_1+\eta(u_2,u_1)} (g(x))^{dx}\right)^{\frac{1}{\eta(u_2,u_1)}} \\ \leq & \, G(f(u_1), f(u_2)) . G(g(u_1), g(u_2)), \end{aligned} \tag{3.4}$$

where $G(.,.)$ is a geometric mean.

(3.4) is called Hermite-Hadamard integral inequalities for the product of multiplicatively preinvex functions.

Proof. Let f and g be positive and multiplicatively preinvex functions and η satisfy Condition C. Then

$$\begin{aligned} & \ln\left(f\left(\frac{2u_1 + \eta(u_2, u_1)}{2}\right) g\left(\frac{2u_1 + \eta(u_2, u_1)}{2}\right)\right) \\ = & \ln\left(f\left(\frac{2u_1 + \eta(u_2, u_1)}{2}\right)\right) + \ln\left(g\left(\frac{2u_1 + \eta(u_2, u_1)}{2}\right)\right) \\ \leq & \ln\left((f(u_1 + t\eta(u_2, u_1)))^{\frac{1}{2}} . (f(u_1 + (1-t)\eta(u_2, u_1)))^{\frac{1}{2}}\right) \\ & + \ln\left((g(u_1 + t\eta(u_2, u_1)))^{\frac{1}{2}} . (g(u_1 + (1-t)\eta(u_2, u_1)))^{\frac{1}{2}}\right) \\ = & \frac{1}{2}\ln(f(u_1 + t\eta(u_2, u_1))) + \frac{1}{2}\ln(f(u_1 + (1-t)\eta(u_2, u_1))) \\ & + \frac{1}{2}\ln(g(u_1 + t\eta(u_2, u_1))) + \frac{1}{2}\ln(g(u_1 + (1-t)\eta(u_2, u_1))). \end{aligned}$$

Integrating the above inequality with respect to t on $[0, 1]$, we have

$$\begin{aligned} & \ln\left(f\left(\frac{2u_1 + \eta(u_2, u_1)}{2}\right) g\left(\frac{2u_1 + \eta(u_2, u_1)}{2}\right)\right) \\ \leq & \int_0^1 \left[\frac{1}{2}\ln(f(u_1 + t\eta(u_2, u_1))) + \frac{1}{2}\ln(f(u_1 + (1-t)\eta(u_2, u_1)))\right] dt \\ & + \int_0^1 \left[\frac{1}{2}\ln(g(u_1 + t\eta(u_2, u_1))) + \frac{1}{2}\ln(g(u_1 + (1-t)\eta(u_2, u_1)))\right] dt \\ = & \frac{1}{2\eta(u_2, u_1)} \int_{u_1}^{u_1+\eta(u_2,u_1)} \ln(f(x))\, dx - \frac{1}{2\eta(u_2, u_1)} \int_{u_1+\eta(u_2,u_1)}^{u_1} \ln(f(x))\, dx \\ & + \frac{1}{2\eta(u_2, u_1)} \int_{u_1}^{u_1+\eta(u_2,u_1)} \ln(g(x))\, dx - \frac{1}{2\eta(u_2, u_1)} \int_{u_1+\eta(u_2,u_1)}^{u_1} \ln(g(x))\, dx \\ = & \frac{1}{\eta(u_2, u_1)} \int_{u_1}^{u_1+\eta(u_2,u_1)} \ln(f(x))\, dx + \frac{1}{\eta(u_2, u_1)} \int_{u_1}^{u_1+\eta(u_2,u_1)} \ln(g(x))\, dx. \end{aligned}$$

Thus, we have

$$
\begin{aligned}
& f\left(\frac{2u_1+\eta(u_2,u_1)}{2}\right)g\left(\frac{2u_1+\eta(u_2,u_1)}{2}\right) \\
\leq\ & e^{\left(\frac{1}{\eta(u_2,u_1)}\int_{u_1}^{u_1+\eta(u_2,u_1)}\ln(f(x))dx+\frac{1}{\eta(u_2,u_1)}\int_{u_1}^{u_1+\eta(u_2,u_1)}\ln(g(x))dx\right)} \\
=\ & \left(e^{\int_{u_1}^{u_1+\eta(u_2,u_1)}\ln(f(x))dx+\int_{u_1}^{u_1+\eta(u_2,u_1)}\ln(g(x))dx}\right)^{\frac{1}{\eta(u_2,u_1)}} \\
=\ & \left(e^{\int_{u_1}^{u_1+\eta(u_2,u_1)}\ln(f(x))dx}.e^{\int_{u_1}^{u_1+\eta(u_2,u_1)}\ln(g(x))dx}\right)^{\frac{1}{\eta(u_2,u_1)}} \\
=\ & \left(\int_{u_1}^{u_1+\eta(u_2,u_1)}(f(x))^{dx}.\int_{u_1}^{u_1+\eta(u_2,u_1)}(g(x))^{dx}\right)^{\frac{1}{\eta(u_2,u_1)}}.
\end{aligned}
$$

Hence,

$$
f\left(\frac{2u_1+\eta(u_2,u_1)}{2}\right)g\left(\frac{2u_1+\eta(u_2,u_1)}{2}\right)\leq\left(\int_{u_1}^{u_1+\eta(u_2,u_1)}(f(x))^{dx}.\int_{u_1}^{u_1+\eta(u_2,u_1)}(g(x))^{dx}\right)^{\frac{1}{\eta(u_2,u_1)}}. \tag{3.5}
$$

Consider the second inequality:

$$
\begin{aligned}
& \left(\int_{u_1}^{u_1+\eta(u_2,u_1)}(f(x))^{dx}.\int_{u_1}^{u_1+\eta(u_2,u_1)}(g(x))^{dx}\right)^{\frac{1}{\eta(u_2,u_1)}} \\
=\ & \left(e^{\int_{u_1}^{u_1+\eta(u_2,u_1)}\ln(f(x))dx+\int_{u_1}^{u_1+\eta(u_2,u_1)}\ln(g(x))dx}\right)^{\frac{1}{\eta(u_2,u_1)}} \\
=\ & \left(e^{\eta(u_2,u_1)\left(\int_0^1\ln(f(u_1+t\eta(u_2,u_1)))dt+\int_0^1\ln(g(u_1+t\eta(u_2,u_1)))dt\right)}\right)^{\frac{1}{\eta(u_2,u_1)}} \\
=\ & e^{\int_0^1\ln(f(u_1+t\eta(u_2,u_1)))dt+\int_0^1\ln(g(u_1+t\eta(u_2,u_1)))dt} \\
\leq\ & e^{\int_0^1\ln\left((f(u_1))^{1-t}(f(u_2))^t\right)dt+\int_0^1\ln\left((g(u_1))^{1-t}(g(u_2))^t\right)dt} \\
=\ & e^{\int_0^1((1-t)\ln f(u_1)+t\ln f(u_2))dt+\int_0^1((1-t)\ln g(u_1)+t\ln g(u_2))dt} \\
=\ & e^{\ln(f(u_1).f(u_2))^{\frac{1}{2}}+\ln(g(u_1).g(u_2))^{\frac{1}{2}}} \\
=\ & \sqrt{f(u_1).f(u_2)}.\sqrt{g(u_1).g(u_2)} \\
=\ & G(f(u_1),f(u_2)).G(g(u_1),g(u_2)).
\end{aligned}
$$

Hence, we have

$$
\left(\int_{u_1}^{u_1+\eta(u_2,u_1)}(f(x))^{dx}.\int_{u_1}^{u_1+\eta(u_2,u_1)}(g(x))^{dx}\right)^{\frac{1}{\eta(u_2,u_1)}}\leq G(f(u_1),f(u_2)).G(g(u_1),g(u_2)). \tag{3.6}
$$

From the inequalities (3.5) and (3.6), we have

$$
f\left(\frac{2u_1+\eta(u_2,u_1)}{2}\right)g\left(\frac{2u_1+\eta(u_2,u_1)}{2}\right)
$$

$$\leq \left(\int_{u_1}^{u_1+\eta(u_2,u_1)} (f(x))^{dx} . \int_{u_1}^{u_1+\eta(u_2,u_1)} (g(x))^{dx}\right)^{\frac{1}{\eta(u_2,u_1)}}$$
$$\leq G(f(u_1), f(u_2)) . G(g(u_1), g(u_2)).$$

□

Theorem 3.3. *Let $\Im \subseteq \mathbb{R}$ an open invex subset with respect to $\eta : \Im \times \Im \to \mathbb{R}$ and $u_1, u_2 \in \Im$ with $u_1 < u_1 + \eta(u_2, u_1)$. If f and g are positive and multiplicatively preinvex functions on the interval $[u_1, u_1 + \eta(u_2, u_1)]$ and η satisfies Condition C, then*

$$\frac{f\left(\frac{2u_1+\eta(u_2,u_1)}{2}\right)}{g\left(\frac{2u_1+\eta(u_2,u_1)}{2}\right)} \leq \left(\frac{\int_{u_1}^{u_1+\eta(u_2,u_1)} (f(x))^{dx}}{\int_{u_1}^{u_1+\eta(u_2,u_1)} (g(x))^{dx}}\right)^{\frac{1}{\eta(u_2,u_1)}} \leq \frac{G(f(u_1), f(u_2))}{G(g(u_1), g(u_2))}, \tag{3.7}$$

where $G(.,.)$ is a geometric mean.

(3.7) is called Hermite-Hadamard integral inequalities for the quotient of multiplicatively preinvex functions.

Proof. Since f and g are positive and multiplicatively preinvex functions and η satisfies Condition C, we can write

$$\begin{aligned}
\ln\frac{f\left(\frac{2u_1+\eta(u_2,u_1)}{2}\right)}{g\left(\frac{2u_1+\eta(u_2,u_1)}{2}\right)} &= \ln\left(f\left(\frac{2u_1+\eta(u_2,u_1)}{2}\right) - g\left(\frac{2u_1+\eta(u_2,u_1)}{2}\right)\right) \\
&\leq \ln\left((f(u_1 + t\eta(u_2,u_1)))^{\frac{1}{2}} . (f(u_1 + (1-t)\eta(u_2,u_1)))^{\frac{1}{2}}\right) \\
&\quad - \ln\left((g(u_1 + t\eta(u_2,u_1)))^{\frac{1}{2}} . (g(u_1 + (1-t)\eta(u_2,u_1)))^{\frac{1}{2}}\right) \\
&= \frac{1}{2}\ln(f(u_1 + t\eta(u_2,u_1))) + \frac{1}{2}\ln(f(u_1 + (1-t)\eta(u_2,u_1))) \\
&\quad -\frac{1}{2}\ln(g(u_1 + t\eta(u_2,u_1))) - \frac{1}{2}\ln(g(u_1 + (1-t)\eta(u_2,u_1))).
\end{aligned}$$

Integrating the above inequality with respect to t on $[0, 1]$, we have

$$\begin{aligned}
&\ln\frac{f\left(\frac{2u_1+\eta(u_2,u_1)}{2}\right)}{g\left(\frac{2u_1+\eta(u_2,u_1)}{2}\right)} \\
\leq &\int_0^1 \left[\frac{1}{2}\ln(f(u_1 + t\eta(u_2,u_1))) + \frac{1}{2}\ln(f(u_1 + (1-t)\eta(u_2,u_1)))\right] dt \\
&- \int_0^1 \left[\frac{1}{2}\ln(g(u_1 + t\eta(u_2,u_1))) + \frac{1}{2}\ln(g(u_1 + (1-t)\eta(u_2,u_1)))\right] dt \\
= &\frac{1}{2\eta(u_2,u_1)}\int_{u_1}^{u_1+\eta(u_2,u_1)} \ln(f(x))\,dx - \frac{1}{2\eta(u_2,u_1)}\int_{u_1+\eta(u_2,u_1)}^{u_1} \ln(f(x))\,dx \\
&-\left[\frac{1}{2\eta(u_2,u_1)}\int_{u_1}^{u_1+\eta(u_2,u_1)} \ln(g(x))\,dx - \frac{1}{2\eta(u_2,u_1)}\int_{u_1+\eta(u_2,u_1)}^{u_1} \ln(g(x))\,dx\right]
\end{aligned}$$

$$= \frac{1}{\eta(u_2,u_1)}\int_{u_1}^{u_1+\eta(u_2,u_1)}\ln(f(x))\,dx - \frac{1}{\eta(u_2,u_1)}\int_{u_1}^{u_1+\eta(u_2,u_1)}\ln(g(x))\,dx.$$

Thus, we have

$$\begin{aligned}
\frac{f\left(\frac{2u_1+\eta(u_2,u_1)}{2}\right)}{g\left(\frac{2u_1+\eta(u_2,u_1)}{2}\right)} &\leq e^{\left(\frac{1}{\eta(u_2,u_1)}\int_{u_1}^{u_1+\eta(u_2,u_1)}\ln(f(x))dx - \frac{1}{\eta(u_2,u_1)}\int_{u_1}^{u_1+\eta(u_2,u_1)}\ln(g(x))dx\right)} \\
&= \left(e^{\int_{u_1}^{u_1+\eta(u_2,u_1)}\ln(f(x))dx - \int_{u_1}^{u_1+\eta(u_2,u_1)}\ln(g(x))dx}\right)^{\frac{1}{\eta(u_2,u_1)}} \\
&= \left(\frac{e^{\int_{u_1}^{u_1+\eta(u_2,u_1)}\ln(f(x))dx}}{e^{\int_{u_1}^{u_1+\eta(u_2,u_1)}\ln(g(x))dx}}\right)^{\frac{1}{\eta(u_2,u_1)}} \\
&= \left(\frac{\int_{u_1}^{u_1+\eta(u_2,u_1)}(f(x))^{dx}}{\int_{u_1}^{u_1+\eta(u_2,u_1)}(g(x))^{dx}}\right)^{\frac{1}{\eta(u_2,u_1)}}.
\end{aligned}$$

Hence,

$$\frac{f\left(\frac{2u_1+\eta(u_2,u_1)}{2}\right)}{g\left(\frac{2u_1+\eta(u_2,u_1)}{2}\right)} \leq \left(\frac{\int_{u_1}^{u_1+\eta(u_2,u_1)}(f(x))^{dx}}{\int_{u_1}^{u_1+\eta(u_2,u_1)}(g(x))^{dx}}\right)^{\frac{1}{\eta(u_2,u_1)}}. \tag{3.8}$$

Now, consider the second inequality

$$\begin{aligned}
&\left(\frac{\int_{u_1}^{u_1+\eta(u_2,u_1)}(f(x))^{dx}}{\int_{u_1}^{u_1+\eta(u_2,u_1)}(g(x))^{dx}}\right)^{\frac{1}{\eta(u_2,u_1)}} \\
&= \left(\frac{e^{\int_{u_1}^{u_1+\eta(u_2,u_1)}\ln(f(x))dx}}{e^{\int_{u_1}^{u_1+\eta(u_2,u_1)}\ln(g(x))dx}}\right)^{\frac{1}{\eta(u_2,u_1)}} \\
&= \left(e^{\int_{u_1}^{u_1+\eta(u_2,u_1)}\ln(f(x))dx - \int_{u_1}^{u_1+\eta(u_2,u_1)}\ln(g(x))dx}\right)^{\frac{1}{\eta(u_2,u_1)}} \\
&= \left(e^{\left(\int_0^1 \ln(f(u_1+t\eta(u_2,u_1)))dt - \int_0^1 \ln(g(u_1+t\eta(u_2,u_1)))dt\right)}\right)^{\frac{1}{\eta(u_2,u_1)}} \\
&= e^{\int_0^1 \ln(f(u_1+t\eta(u_2,u_1)))dt - \int_0^1 \ln(g(u_1+t\eta(u_2,u_1)))dt} \\
&\leq e^{\int_0^1 \ln\left((f(u_1))^{1-t}(f(u_2))^t\right)dt - \int_0^1 \ln\left((g(u_1))^{1-t}(g(u_2))^t\right)dt} \\
&= e^{\int_0^1((1-t)\ln f(u_1)+t\ln f(u_2))dt - \int_0^1((1-t)\ln g(u_1)+t\ln g(u_2))dt} \\
&= e^{\ln(f(u_1).f(u_2))^{\frac{1}{2}} - \ln(g(u_1).g(u_2))^{\frac{1}{2}}} \\
&= \frac{\sqrt{f(u_1).f(u_2)}}{\sqrt{g(u_1).g(u_2)}} \\
&= \frac{G(f(u_1),f(u_2))}{G(g(u_1),g(u_2))}.
\end{aligned}$$

Hence,

$$\left(\frac{\int_{u_1}^{u_1+\eta(u_2,u_1)}(f(x))^{dx}}{\int_{u_1}^{u_1+\eta(u_2,u_1)}(g(x))^{dx}}\right)^{\frac{1}{\eta(u_2,u_1)}} \leq \frac{G(f(u_1),f(u_2))}{G(g(u_1),g(u_2))}. \tag{3.9}$$

Combining the inequalities (3.8) and (3.9), we have

$$\frac{f\left(\frac{2u_1+\eta(u_2,u_1)}{2}\right)}{g\left(\frac{2u_1+\eta(u_2,u_1)}{2}\right)} \leq \left(\frac{\int_{u_1}^{u_1+\eta(u_2,u_1)}(f(x))^{dx}}{\int_{u_1}^{u_1+\eta(u_2,u_1)}(g(x))^{dx}}\right)^{\frac{1}{\eta(u_2,u_1)}} \leq \frac{G(f(u_1),f(u_2))}{G(g(u_1),g(u_2))}.$$

□

Theorem 3.4. *Let $\mathfrak{I} \subseteq \mathbb{R}$ an open invex subset with respect to $\eta : \mathfrak{I} \times \mathfrak{I} \to \mathbb{R}$ and $u_1, u_2 \in \mathfrak{I}$ with $u_1 < u_1 + \eta(u_2, u_1)$. Let f and g be preinvex and multiplicatively preinvex positive functions, respectively, on the interval $[u_1, u_1 + \eta(u_2, u_1)]$. Then, we have*

$$\left(\frac{\int_{u_1}^{u_1+\eta(u_2,u_1)}(f(x))^{dx}}{\int_{u_1}^{u_1+\eta(u_2,u_1)}(g(x))^{dx}}\right)^{\frac{1}{\eta(u_2,u_1)}} \leq \frac{\left(\frac{(f(u_2))^{f(u_2)}}{(f(u_1))^{f(u_1)}}\right)^{\frac{1}{f(u_2)-f(u_1)}}}{e.G(g(u_1),g(u_2))},$$

where $G(.,.)$ is a geometric mean.

Proof. Note that,

$$\begin{aligned}
&\left(\frac{\int_{u_1}^{u_1+\eta(u_2,u_1)}(f(x))^{dx}}{\int_{u_1}^{u_1+\eta(u_2,u_1)}(g(x))^{dx}}\right)^{\frac{1}{\eta(u_2,u_1)}} \\
=&\left(\frac{e^{\int_{u_1}^{u_1+\eta(u_2,u_1)}\ln(f(x))dx}}{e^{\int_{u_1}^{u_1+\eta(u_2,u_1)}\ln(g(x))dx}}\right)^{\frac{1}{\eta(u_2,u_1)}} \\
=&\left(e^{\int_{u_1}^{u_1+\eta(u_2,u_1)}\ln(f(x))dx-\int_{u_1}^{u_1+\eta(u_2,u_1)}\ln(g(x))dx}\right)^{\frac{1}{\eta(u_2,u_1)}} \\
=&\; e^{\int_0^1\ln(f(u_1+t\eta(u_2,u_1)))dt-\int_0^1\ln(g(u_1+t\eta(u_2,u_1)))dt} \\
\leq&\; e^{\int_0^1\ln(f(u_1)+t(f(u_2)-f(u_1)))dt-\int_0^1\ln\left((g(u_1))^{1-t}(g(u_2))^{t}\right)dt} \\
=&\; e^{\ln\left(\left(\frac{(f(u_2))^{f(u_2)}}{(f(u_1))^{f(u_1)}}\right)^{\frac{1}{f(u_2)-f(u_1)}}\right)-1-\ln(G(g(u_1),g(u_2)))} \\
=&\;\frac{\left(\frac{(f(u_2))^{f(u_2)}}{(f(u_1))^{f(u_1)}}\right)^{\frac{1}{f(u_2)-f(u_1)}}}{e.G(g(u_1),g(u_2))}.
\end{aligned}$$

Thus, we have

$$\left(\frac{\int_{u_1}^{u_1+\eta(u_2,u_1)}(f(x))^{dx}}{\int_{u_1}^{u_1+\eta(u_2,u_1)}(g(x))^{dx}}\right)^{\frac{1}{\eta(u_2,u_1)}} \leq \frac{\left(\frac{(f(u_2))^{f(u_2)}}{(f(u_1))^{f(u_1)}}\right)^{\frac{1}{f(u_2)-f(u_1)}}}{e.G(g(u_1),g(u_2))},$$

which completes the proof. □

Theorem 3.5. *Let $\Im \subseteq \mathbb{R}$ an open invex subset with respect to $\eta : \Im \times \Im \to \mathbb{R}$ and $u_1, u_2 \in \Im$ with $u_1 < u_1+\eta(u_2, u_1)$ .Let f and g be multiplicatively preinvex and preinvex positive functions,respectively, on the interval $[u_1, u_1 + \eta(u_2, u_1)]$. Then, we have*

$$\left(\frac{\int_{u_1}^{u_1+\eta(u_2,u_1)} (f(x))^{dx}}{\int_{u_1}^{u_1+\eta(u_2,u_1)} (g(x))^{dx}}\right)^{\frac{1}{\eta(u_2,u_1)}} \leq \frac{e.G(f(u_1), f(u_2))}{\left(\frac{(g(u_2))^{g(u_2)}}{(g(u_1))^{g(u_1)}}\right)^{\frac{1}{g(u_2)-g(u_1)}}},$$

where $G(.,.)$ is a geometric mean.

Proof. Note that

$$\begin{aligned}
&\left(\frac{\int_{u_1}^{u_1+\eta(u_2,u_1)} (f(x))^{dx}}{\int_{u_1}^{u_1+\eta(u_2,u_1)} (g(x))^{dx}}\right)^{\frac{1}{\eta(u_2,u_1)}} \\
= &\left(\frac{e^{\int_{u_1}^{u_1+\eta(u_2,u_1)} \ln(f(x))dx}}{e^{\int_{u_1}^{u_1+\eta(u_2,u_1)} \ln(g(x))dx}}\right)^{\frac{1}{\eta(u_2,u_1)}} \\
= &\left(e^{\int_{u_1}^{u_1+\eta(u_2,u_1)} \ln(f(x))dx-\int_{u_1}^{u_1+\eta(u_2,u_1)} \ln(g(x))dx}\right)^{\frac{1}{\eta(u_2,u_1)}} \\
= &\ e^{\int_0^1 \ln(f(u_1+t\eta(u_2,u_1)))dt-\int_0^1 \ln(g(u_1+t\eta(u_2,u_1)))dt} \\
\leq &\ e^{\int_0^1 \ln\left((f(u_1))^{1-t}(f(u_2))^t\right)dt-\int_0^1 \ln(g(u_1)+t(g(u_2)-g(u_1)))dt} \\
= &\ e^{\ln(G(f(u_1),f(u_2)))-\ln\left(\left(\frac{(g(u_2))^{g(u_2)}}{(g(u_1))^{g(u_1)}}\right)^{\frac{1}{g(u_2)-g(u_1)}}\right)+1} \\
= &\ \frac{e.G(f(u_1), f(u_2))}{\left(\frac{(g(u_2))^{g(u_2)}}{(g(u_1))^{g(u_1)}}\right)^{\frac{1}{g(u_2)-g(u_1)}}}.
\end{aligned}$$

Hence,

$$\left(\frac{\int_{u_1}^{u_1+\eta(u_2,u_1)} (f(x))^{dx}}{\int_{u_1}^{u_1+\eta(u_2,u_1)} (g(x))^{dx}}\right)^{\frac{1}{\eta(u_2,u_1)}} \leq \frac{e.G(f(u_1), f(u_2))}{\left(\frac{(g(u_2))^{g(u_2)}}{(g(u_1))^{g(u_1)}}\right)^{\frac{1}{g(u_2)-g(u_1)}}},$$

which is the desired result.

□

Theorem 3.6. *Let $\Im \subseteq \mathbb{R}$ an open invex subset with respect to $\eta : \Im \times \Im \to \mathbb{R}$ and $u_1, u_2 \in \Im$ with $u_1 < u_1+\eta(u_2, u_1)$. Let f and g be preinvex and multiplicatively preinvex positive functions,respectively, on the interval $[u_1, u_1 + \eta(u_2, u_1)]$. Then, we have*

$$\left(\int_{u_1}^{u_1+\eta(u_2,u_1)} (f(x))^{dx} . \int_{u_1}^{u_1+\eta(u_2,u_1)} (g(x))^{dx}\right)^{\frac{1}{\eta(u_2,u_1)}} \leq \frac{\left(\frac{(f(u_2))^{f(u_2)}}{(f(u_1))^{f(u_1)}}\right)^{\frac{1}{f(u_2)-f(u_1)}} .G(g(u_1), g(u_2))}{e},$$

where $G(.,.)$ is a geometric mean.

Proof. Note that

$$
\begin{aligned}
&\left(\int_{u_1}^{u_1+\eta(u_2,u_1)}(f(x))^{dx}\cdot\int_{u_1}^{u_1+\eta(u_2,u_1)}(g(x))^{dx}\right)^{\frac{1}{\eta(u_2,u_1)}}\\
=&\left(e^{\int_{u_1}^{u_1+\eta(u_2,u_1)}\ln(f(x))dx+\int_{u_1}^{u_1+\eta(u_2,u_1)}\ln(g(x))dx}\right)^{\frac{1}{\eta(u_2,u_1)}}\\
=&\left(e^{\eta(u_2,u_1)\left(\int_0^1\ln(f(u_1+t\eta(u_2,u_1)))dt+\int_0^1\ln(g(u_1+t\eta(u_2,u_1)))dt\right)}\right)^{\frac{1}{\eta(u_2,u_1)}}\\
=&\ e^{\int_0^1\ln(f(u_1+t\eta(u_2,u_1)))dt+\int_0^1\ln(g(u_1+t\eta(u_2,u_1)))dt}\\
\leq&\ e^{\int_0^1\ln(f(u_1)+t(f(u_2)-f(u_1)))dt-\int_0^1\ln\left((g(u_1))^{1-t}(g(u_2))^t\right)dt}\\
=&\ e^{\ln\left(\left(\frac{(f(u_2))^{f(u_2)}}{(f(u_1))^{f(u_1)}}\right)^{\frac{1}{f(u_2)-f(u_1)}}\right)-1+\ln(G(g(u_1),g(u_2)))}\\
=&\ \frac{\left(\frac{(f(u_2))^{f(u_2)}}{(f(u_1))^{f(u_1)}}\right)^{\frac{1}{f(u_2)-f(u_1)}}.G\left(g\left(u_1\right),g\left(u_2\right)\right)}{e}.
\end{aligned}
$$

Hence,

$$
\left(\int_{u_1}^{u_1+\eta(u_2,u_1)}(f(x))^{dx}\cdot\int_{u_1}^{u_1+\eta(u_2,u_1)}(g(x))^{dx}\right)^{\frac{1}{\eta(u_2,u_1)}}\leq\frac{\left(\frac{(f(u_2))^{f(u_2)}}{(f(u_1))^{f(u_1)}}\right)^{\frac{1}{f(u_2)-f(u_1)}}.G\left(g\left(u_1\right),g\left(u_2\right)\right)}{e}.
$$

This completes the proof. □

Acknowledgements

This research article is supported by Kırklareli University Scientific Research Projects Coordination Unit. Project Number: KLUBAP-191.

Conflict of interest

The author declares no conflict of interest in this paper.

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