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On Refinements of Some Integral Inequalities for Differentiable Prequasiinvex Functions

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Abstract. In this paper, using the new and improved form of Hölder's integral inequality called Hölder-İşcan integral inequality, some new inequalities of the right-hand side of Hermite-Hadamard type inequality for prequasiinvex functions are established. The results obtained are compared with the known results. It is shown that the results obtained in this paper are better than those known ones.

1. Introduction

The concept of convexity has become a deep research area in pure and applied sciences. In recent years, several extensions and generalizations of classical convexity have been studied by many researchers using novel methods and ideas. Hanson [6] introduced an important generalization of convex functions called invex functions. Ben-Israel and Mond [4] introduced the notions of invex sets and preinvex functions. For recent applications and generalizations of the preinvex functions, we refer the interested reader to [2, 3, 9, 11, 14, 17].

Inequalities play a fundamental role in many branches of pure and applied mathematics. A number of studies have shown that convexity has a closely relationship with the theory of inequalities. One of the most famous inequality for convex functions is named Hermite-Hadamard integral inequality as follows:

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)\, dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

Recently, Hermite-Hadamard inequality for convex functions and their variant forms has been a rich sourge of inspiration. For recent results, improvements, extensions and generalizations the inequality (1), please refer the monographs [1, 5, 7, 14, 15, 17, 19].

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2. Preliminaries

Definition 2.1. *[20] Let K be a closed set in $\mathbb{R}^n$. Suppose that $f : K \to \mathbb{R}$ and $\eta : K \times K \to \mathbb{R}$ be continuous functions. Let $u \in K$, then the set K is said to be invex at each u with respect to $\eta(.,.)$, if*

$$u + t\eta(v,u) \in K, \forall u, v \in K, t \in [0,1].$$

K is said to be an invex set with respect to η, if K is invex at each $u \in K$. The invex set K is also called η-connected set.

Note that if $\eta(v,u) = v - u$, invexity reduces to convexity. Thus, every convex set is also an invex set with respect to $\eta(v,u) = v - u$, but the converse is not true in general.

Definition 2.2. *[18] The function f on the invex set K is said to be preinvex with respect to η, if the inequality*

$$f(u + t\eta(v,u)) \leq (1-t) f(u) + t f(v)$$

holds for all $u, v \in K$ and $t \in [0,1]$.

Definition 2.3. *[16] The function f on the invex set K is said to be prequasiinvex with respect to η, if*

$$f(u + t\eta(v,u)) \leq \max\{f(u), f(v)\}$$

for all $u, v \in K$ and $t \in [0,1]$.

Recently, Noor [14] has obtained the new form of Hermite-Hadamard inequality for the preinvex functions:

Theorem 2.4. *[14] Let $f : [a, a+\eta(b,a)] \to (0,\infty)$ be a preinvex function on the interval of the real numbers K° (interior of K) and $a, b \in K^\circ$ with $a < a + \eta(b,a)$. Then the following inequality holds:*

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \leq \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{2}$$

The famous Hölder integral inequality is given as follows:

Theorem 2.5. *[13] Let $p > 1$ and $1/p + 1/q = 1$. If f and g are real functions defined on $[a,b]$ and if $|f|^p$ and $|g|^q$ are integrable functions on $[a,b]$, then*

$$\int_a^b |f(x)g(x)|\,dx \leq \left(\int_a^b |f(x)|^p\,dx\right)^{\frac{1}{p}} \left(\int_a^b |g(x)|^q\,dx\right)^{\frac{1}{q}},$$

with equality holding if and only if $A|f(x)|^p = B|g(x)|^q$ almost everywhere, where A and B are constants.

Power-mean integral inequality as a different version of Hölder integral inequality can be given as follows:

Theorem 2.6. *[13] Let $q \geq 1$. If f and g are real functions defined on $[a,b]$ and if $|f|$, $|f||g|^q$ are integrable functions on $[a,b]$, then*

$$\int_a^b |f(x)g(x)|\,dx \leq \left(\int_a^b |f(x)|\,dx\right)^{1-\frac{1}{q}} \left(\int_a^b |f(x)||g(x)|^q\,dx\right)^{\frac{1}{q}}.$$

In [12], Latif obtained some inequalities of Hermite-Hadamard type for differentiable prequasiinvex mappings connected with the right part of the inequality (2) were proved using the following lemma:

Lemma 2.7. *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose that $f : K \to \mathbb{R}$ is a differentiable mapping on K such that $f' \in L([a, a + \eta(b, a)])$. Then for all $a, b \in K$ with $a < a + \eta(b, a)$ the following equality holds:*

$$\begin{aligned} & \frac{f(a) + f(a + \eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx \\ = & \frac{\eta(b,a)}{4} \left[\int_0^1 (-t) f'\left(a + \left(\frac{1-t}{2}\right)\eta(b,a)\right) dt + \int_0^1 t f'\left(a + \left(\frac{1+t}{2}\right)\eta(b,a)\right) dt \right]. \end{aligned}$$

In [12], Latif obtained the following result for prequasiinvex functions using the above lemma:

Theorem 2.8. *Let $K \subseteq [0, \infty)$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and $a, b \in K$ with $a < a + \eta(b, a)$. Assume that $f : K \to \mathbb{R}$ is a differentiable mapping on K such that $f' \in L([a, a + \eta(b, a)])$. If $|f'|$ is prequasiinvex on K, then for every $a, b \in K$ with $\eta(b, a) > 0$ we have the following inequality:*

$$\begin{aligned} & \left| \frac{f(a) + f(a + \eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx \right| \\ \le & \frac{\eta(b,a)}{8} \left[\sup\left\{ |f'(a)|, \left| f'\left(a + \frac{1}{2}\eta(b,a)\right) \right| \right\} + \sup\left\{ \left| f'\left(a + \frac{1}{2}\eta(b,a)\right) \right|, |f'(a + \eta(b,a))| \right\} \right]. \end{aligned} \tag{3}$$

In [12], Latif obtained the following result for prequasiinvex functions using Lemma 2.7 and Hölder integral inequality:

Theorem 2.9. *Let $K \subseteq [0, \infty)$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and $a, b \in K$ with $a < a + \eta(b, a)$. Assume that $f : K \to \mathbb{R}$ is a differentiable mapping on K such that $f' \in L([a, a + \eta(b, a)])$. If $|f'|^q$ is prequasiinvex on K for $q > 1$, then, for every $a, b \in K$ with $\eta(b, a) > 0$ we have the following inequality:*

$$\begin{aligned} & \left| \frac{f(a) + f(a + \eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx \right| \\ \le & \frac{\eta(b,a)}{4(p+1)^{1/p}} \left[\left(\sup\left\{ |f'(a)|^q, \left| f'\left(a + \frac{1}{2}\eta(b,a)\right) \right|^q \right\} \right)^{1/q} \right. \\ & \left. + \left(\sup\left\{ \left| f'\left(a + \frac{1}{2}\eta(b,a)\right) \right|^q, |f'(a + \eta(b,a))|^q \right\} \right)^{1/q} \right], \end{aligned} \tag{4}$$

where $1/p + 1/q = 1$.

In [12], Latif also obtained the following result for prequasiinvex functions using Lemma 2.7 and power-mean integral inequality:

Theorem 2.10. *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and $a, b \in K$ with $a < a + \eta(b, a)$. Suppose $f : K \to \mathbb{R}$ is a differentiable mapping on K such that $f' \in L([a, a + \eta(b, a)])$. If $|f'|^q$ is prequasiinvex on K for $q \ge 1$, then for every $a, b \in K$ with $\eta(b, a) > 0$ the following inequality holds:*

$$\begin{aligned} & \left| \frac{f(a) + f(a + \eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx \right| \\ \le & \frac{\eta(b,a)}{8} \left[\left(\sup\left\{ |f'(a)|^q, \left| f'\left(a + \frac{1}{2}\eta(b,a)\right) \right|^q \right\} \right)^{1/q} + \left(\sup\left\{ \left| f'\left(a + \frac{1}{2}\eta(b,a)\right) \right|^q, |f'(a + \eta(b,a))|^q \right\} \right)^{1/q} \right]. \end{aligned} \tag{5}$$

In [8], İşcan obtained the following integral inequality which gives better results than classical Hölder inequality:

Theorem 2.11. *(Hölder-İşcan Integral Inequality) Let $p > 1$ and $1/p + 1/q = 1$. If f and g are real functions defined on $[a, b]$ and if $|f|^p$ and $|g|^q$ are integrable functions on $[a, b]$, then*

$$i) \quad \int_a^b |f(x) g(x)| \, dx \le \frac{1}{b-a} \left\{ \left(\int_a^b (b-x) |f(x)|^p \, dx \right)^{\frac{1}{p}} \left(\int_a^b (b-x) |g(x)|^q \, dx \right)^{\frac{1}{q}} + \left(\int_a^b (x-a) |f(x)|^p \, dx \right)^{\frac{1}{p}} \left(\int_a^b (x-a) |g(x)|^q \, dx \right)^{\frac{1}{q}} \right\} \tag{6}$$

$$ii) \quad \frac{1}{b-a} \left\{ \left(\int_a^b (b-x) |f(x)|^p \, dx \right)^{\frac{1}{p}} \left(\int_a^b (b-x) |g(x)|^q \, dx \right)^{\frac{1}{q}} + \left(\int_a^b (x-a) |f(x)|^p \, dx \right)^{\frac{1}{p}} \left(\int_a^b (x-a) |g(x)|^q \, dx \right)^{\frac{1}{q}} \right\} \le \left(\int_a^b |f(x)|^p \, dx \right)^{\frac{1}{p}} \left(\int_a^b |g(x)|^q \, dx \right)^{\frac{1}{q}}$$

In [10], a different represent of Hölder-İşcan inequality was given as follows:

Theorem 2.12. *(Improved Power-mean Integral Inequality) Let $q \ge 1$. If f and g are real functions defined on $[a, b]$ and if $|f|$, $|f| |g|^q$ are integrable functions on $[a, b]$, then*

$$i) \quad \int_a^b |f(x) g(x)| \, dx \le \frac{1}{b-a} \left\{ \left(\int_a^b (b-x) |f(x)| \, dx \right)^{1-\frac{1}{q}} \left(\int_a^b (b-x) |f(x)| |g(x)|^q \, dx \right)^{\frac{1}{q}} + \left(\int_a^b (x-a) |f(x)| \, dx \right)^{1-\frac{1}{q}} \left(\int_a^b (x-a) |f(x)| |g(x)|^q \, dx \right)^{\frac{1}{q}} \right\} \tag{7}$$

$$ii) \quad \frac{1}{b-a} \left\{ \left(\int_a^b (b-x) |f(x)| \, dx \right)^{1-\frac{1}{q}} \left(\int_a^b (b-x) |f(x)| |g(x)|^q \, dx \right)^{\frac{1}{q}} + \left(\int_a^b (x-a) |f(x)| \, dx \right)^{1-\frac{1}{q}} \left(\int_a^b (x-a) |f(x)| |g(x)|^q \, dx \right)^{\frac{1}{q}} \right\} \le \left(\int_a^b |f(x)| \, dx \right)^{1-\frac{1}{q}} \left(\int_a^b |f(x)| |g(x)|^q \, dx \right)^{\frac{1}{q}}$$

3. Main Results

In this section, we will obtain some new upper bounds for the right-hand side of Hermite-Hadamard inequality for differentiable prequasiinvex functions and we will show that the new upper bounds we obtained are better than the ones given in [12].

Theorem 3.1. *Let $K \subseteq [0, \infty)$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and $a, b \in K$ with $a < a + \eta(b, a)$. Assume that $f : K \to \mathbb{R}$ is a differentiable mapping on K such that $f' \in L([a, a + \eta(b, a)])$. If $|f'|^q$ is prequasiinvex on K for $q > 1$, then, for every $a, b \in K$ with $\eta(b, a) > 0$ we have the following inequality:*

$$\left| \frac{f(a) + f(a + \eta(b, a))}{2} - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(x) \, dx \right| \le \frac{\eta(b, a)}{4} \left(\frac{1}{2} \right)^{1/q} \left(\frac{1}{p+2} \right)^{1/p} \left[\left(\frac{1}{p+1} \right)^{1/p} + 1 \right] \times \left[\left(\sup \left\{ |f'(a)|^q, \left| f'\left(a + \frac{1}{2}\eta(b, a)\right) \right|^q \right\} \right)^{1/q} + \left(\sup \left\{ \left| f'\left(a + \frac{1}{2}\eta(b, a)\right) \right|^q, |f'(a + \eta(b, a))|^q \right\} \right)^{1/q} \right], \tag{8}$$

where $1/p + 1/q = 1$.

Proof. From Lemma 2.7 and using Hölder-İşcan integral inequality (6), we have

$$\begin{aligned}
&\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right| \\
\leq\ & \frac{\eta(b,a)}{4}\left[\int_0^1 |-t|\left|f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|dt+\int_0^1 |t|\left|f'\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|dt\right] \\
\leq\ & \frac{\eta(b,a)}{4}\left[\left(\int_0^1 (1-t)|-t|^p\,dt\right)^{1/p}\left(\int_0^1 (1-t)\left|f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|^q dt\right)^{1/q}\right. \\
&\left.+\left(\int_0^1 t|-t|^p\,dt\right)^{1/p}\left(\int_0^1 t\left|f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|^q dt\right)^{1/q}\right] \\
&+\frac{\eta(b,a)}{4}\left[\left(\int_0^1 (1-t)|t|^p\,dt\right)^{1/p}\left(\int_0^1 (1-t)\left|f'\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|^q dt\right)^{1/q}\right. \\
&\left.+\left(\int_0^1 t|t|^p\,dt\right)^{1/p}\left(\int_0^1 t\left|f'\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|^q dt\right)^{1/q}\right].
\end{aligned}$$

By the prequasiinvexity of $|f'|^q$ on K for $q>1$, then, for every $a,b\in K$ with $\eta(b,a)>0$ and $t\in[0,1]$, we have

$$\left|f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|^q \leq \sup\left\{|f'(a)|^q, \left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|^q\right\}, \tag{9}$$

and

$$\left|f'\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|^q \leq \sup\left\{\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|^q, |f'(a+\eta(b,a))|^q\right\}, \tag{10}$$

where $1/p + 1/q = 1$.

Using the inequalities (9) and (10) we have,

$$\begin{aligned}
&\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right| \\
\leq\ & \frac{\eta(b,a)}{4}\left(\frac{1}{2}\right)^{1/q}\left[\left(\frac{1}{(p+1)(p+2)}\right)^{1/p}+\left(\frac{1}{p+2}\right)^{1/p}\right]\left(\sup\left\{|f'(a)|^q, \left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|^q\right\}\right)^{1/q} \\
&+\frac{\eta(b,a)}{4}\left(\frac{1}{2}\right)^{1/q}\left[\left(\frac{1}{(p+1)(p+2)}\right)^{1/p}+\left(\frac{1}{p+2}\right)^{1/p}\right]\left(\sup\left\{\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|^q, |f'(a+\eta(b,a))|^q\right\}\right)^{1/q} \\
=\ & \frac{\eta(b,a)}{4}\left(\frac{1}{2}\right)^{1/q}\left(\frac{1}{p+2}\right)^{1/p}\left[\left(\frac{1}{p+1}\right)^{1/p}+1\right] \\
&\times\left[\left(\sup\left\{|f'(a)|^q, \left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|^q\right\}\right)^{1/q}+\left(\sup\left\{\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|^q, |f'(a+\eta(b,a))|^q\right\}\right)^{1/q}\right]
\end{aligned}$$

where

$$\int_0^1 t\,dt = \int_0^1 (1-t)\,dt = \frac{1}{2},$$

$$\int_0^1 t\,|t|^p\,dt = \frac{1}{p+2}$$

and

$$\int_0^1 (1-t)\,|t|^p\,dt = \frac{1}{(p+1)(p+2)}.$$

□

Remark 3.2. *The inequality (8) gives better results than the inequality (4). Let us show that*

$$\left(\frac{1}{2}\right)^{1/q}\left(\frac{1}{p+2}\right)^{1/p}\left[\left(\frac{1}{p+1}\right)^{1/p}+1\right] \le \left(\frac{1}{p+1}\right)^{1/p}.$$

An easy calculation gives

$$\frac{\left(\frac{1}{2}\right)^{1/q}\left(\frac{1}{p+2}\right)^{1/p}\left[\left(\frac{1}{p+1}\right)^{1/p}+1\right]}{\left(\frac{1}{p+1}\right)^{1/p}} = \left(\frac{1}{2}\right)^{1/q}\left[\left(\frac{1}{p+2}\right)^{1/p}+\left(\frac{p+1}{p+2}\right)^{1/p}\right].$$

Thus, using concavity of the function $h:[0,\infty)\to\mathbb{R}$, $h(x)=x^s$, $0<s\le 1$, we have

$$\begin{aligned}
\left(\frac{1}{2}\right)^{1/q}\left[\left(\frac{1}{p+2}\right)^{1/p}+\left(\frac{p+1}{p+2}\right)^{1/p}\right] &= 2^{1/p}\left[\frac{1}{2}\left(\frac{1}{p+2}\right)^{1/p}+\frac{1}{2}\left(\frac{p+1}{p+2}\right)^{1/p}\right] \\
&\le 2^{1/p}\left(\frac{\frac{1}{p+2}+\frac{p+1}{p+2}}{2}\right)^{1/p} \\
&= 1
\end{aligned}$$

which is the required.

Theorem 3.3. *Let $K\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta: K\times K\to\mathbb{R}$ and $a,b\in K$ with $a<a+\eta(b,a)$. Suppose $f:K\to\mathbb{R}$ is a differentiable mapping on K such that $f'\in L([a,a+\eta(b,a)])$. If $|f'|^q$ is prequasiinvex on K for $q\ge 1$, then for every $a,b\in K$ with $\eta(b,a)>0$ the following inequality holds:*

$$\begin{aligned}
&\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \\
\le\ & \frac{\eta(b,a)}{4}\left(\frac{1}{q+1}\right)^{1/q}\left[\left(\sup\left\{|f'(a)|^q,\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|^q\right\}\right)^{1/q}\right. \\
&\left.+\left(\sup\left\{\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|^q,|f'(a+\eta(b,a))|^q\right\}\right)^{1/q}\right]. \qquad (11)
\end{aligned}$$

Proof. Using Lemma 2.7, power-mean integral inequality and prequasiinvexity of $|f'|^q$, we get

$$\begin{aligned}
&\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \\
\le\ & \frac{\eta(b,a)}{4}\left[\int_0^1 |-t|\left|f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|dt+\int_0^1 |t|\left|f'\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|dt\right]
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\eta(b,a)}{4}\left[\left(\int_0^1 dt\right)^{1-1/q}\left(\int_0^1 |-t|^q \left|f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|^q dt\right)^{1/q}\right] \\
&\quad +\frac{\eta(b,a)}{4}\left[\left(\int_0^1 dt\right)^{1-1/q}\left(\int_0^1 |t|^q \left|f'\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|^q dt\right)^{1/q}\right] \\
&\leq \frac{\eta(b,a)}{4}\left(\frac{1}{q+1}\right)^{1/q}\left(\sup\left\{\left|f'(a)\right|^q,\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|^q\right\}\right)^{1/q} \\
&\quad +\frac{\eta(b,a)}{4}\left(\frac{1}{q+1}\right)^{1/q}\left(\sup\left\{\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|^q,\left|f'(a+\eta(b,a))\right|^q\right\}\right)^{1/q} \\
&= \frac{\eta(b,a)}{4}\left(\frac{1}{q+1}\right)^{1/q}\left[\left(\sup\left\{\left|f'(a)\right|^q,\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|^q\right\}\right)^{1/q}\right. \\
&\quad \left.+\left(\sup\left\{\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|^q,\left|f'(a+\eta(b,a))\right|^q\right\}\right)^{1/q}\right],
\end{aligned}
$$

where

$$\int_0^1 |t|^q \, dt = \frac{1}{q+1}.$$

□

If $q = 1$ in the inequality (11), then we get the following result:

Corollary 3.4. *Let $K \subseteq [0,\infty)$ be an open invex subset with respect to $\eta : K\times K \to \mathbb{R}$ and $a,b \in K$ with $a < a+\eta(b,a)$. Assume that $f : K \to \mathbb{R}$ is a differentiable mapping on K such that $f' \in L([a, a+\eta(b,a)])$. If $|f'|$ is prequasiinvex on K, then for every $a,b \in K$ with $\eta(b,a) > 0$ we have the following inequality:*

$$
\begin{aligned}
&\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \\
&\leq \frac{\eta(b,a)}{8}\left[\sup\left\{\left|f'(a)\right|,\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|\right\}+\sup\left\{\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|,\left|f'(a+\eta(b,a))\right|\right\}\right]. \qquad (12)
\end{aligned}
$$

Note that, the inequality (12) coincides with the inequality (3) in Theorem 2.8.

Theorem 3.5. *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K\times K \to \mathbb{R}$ and $a,b \in K$ with $a < a+\eta(b,a)$. Suppose $f : K \to \mathbb{R}$ is a differentiable mapping on K such that $f' \in L([a, a+\eta(b,a)])$. If $|f'|^q$ is prequasiinvex on K for $q \geq 1$, then for every $a,b \in K$ with $\eta(b,a) > 0$ the following inequality holds:*

$$
\begin{aligned}
&\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \leq \frac{\eta(b,a)}{4}\left(\frac{1}{2}\right)^{1-1/q}\left(\frac{1}{q+2}\right)^{1/q}\left[\left(\frac{1}{q+1}\right)^{1/q}+1\right] \\
&\times\left[\left(\sup\left\{\left|f'(a)\right|^q,\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|^q\right\}\right)^{1/q}+\left(\sup\left\{\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|^q,\left|f'(a+\eta(b,a))\right|^q\right\}\right)^{1/q}\right]. \qquad (13)
\end{aligned}
$$

Proof. From Lemma 2.7, improved power-mean integral inequality (7) and prequasiinvexity of $|f'|^q$, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \\
\leq\ & \frac{\eta(b,a)}{4}\left[\int_0^1 |-t|\left|f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|dt+\int_0^1 |t|\left|f'\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|dt\right] \\
\leq\ & \frac{\eta(b,a)}{4}\Bigg[\left(\int_0^1 (1-t)\,dt\right)^{1-1/q}\left(\int_0^1 (1-t)\,|-t|^q\left|f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|^q dt\right)^{1/q} \\
&+\left(\int_0^1 tdt\right)^{1-1/q}\left(\int_0^1 t\,|-t|^q\left|f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|^q dt\right)^{1/q}\Bigg] \\
&+\frac{\eta(b,a)}{4}\Bigg[\left(\int_0^1 (1-t)\,dt\right)^{1-1/q}\left(\int_0^1 (1-t)\,|t|^q\left|f'\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|^q dt\right)^{1/q} \\
&+\left(\int_0^1 tdt\right)^{1-1/q}\left(\int_0^1 t\,|t|^q\left|f'\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|^q dt\right)^{1/q}\Bigg] \\
\leq\ & \frac{\eta(b,a)}{4}\left(\frac{1}{2}\right)^{1-1/q}\left(\frac{1}{q+2}\right)^{1/q}\left[\left(\frac{1}{q+1}\right)^{1/q}+1\right]\left(\sup\left\{\left|f'(a)\right|^q,\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|^q\right\}\right)^{1/q} \\
&+\frac{\eta(b,a)}{4}\left(\frac{1}{2}\right)^{1-1/q}\left(\frac{1}{q+2}\right)^{1/q}\left[\left(\frac{1}{q+1}\right)^{1/q}+1\right]\left(\sup\left\{\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|^q,\left|f'(a+\eta(b,a))\right|^q\right\}\right)^{1/q} \\
=\ & \frac{\eta(b,a)}{4}\left(\frac{1}{2}\right)^{1-1/q}\left(\frac{1}{q+2}\right)^{1/q}\left[\left(\frac{1}{q+1}\right)^{1/q}+1\right] \\
&\times\left[\left(\sup\left\{\left|f'(a)\right|^q,\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|^q\right\}\right)^{1/q}+\left(\sup\left\{\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|^q,\left|f'(a+\eta(b,a))\right|^q\right\}\right)^{1/q}\right],
\end{aligned}$$

where

$$\int_0^1 (1-t)\,|t|^q\,dt=\frac{1}{(q+1)(q+2)}$$

and

$$\int_0^1 t\,|t|^q\,dt=\frac{1}{q+2}.$$

□

If $q=1$ in the inequality (13), then we get the following result:

Corollary 3.6. *Let $K\subseteq[0,\infty)$ be an open invex subset with respect to $\eta: K\times K\to\mathbb{R}$ and $a,b\in K$ with $a<a+\eta(b,a)$. Assume that $f:K\to\mathbb{R}$ is a differentiable mapping on K such that $f'\in L([a,a+\eta(b,a)])$. If $\left|f'\right|$ is prequasiinvex on K, then for every $a,b\in K$ with $\eta(b,a)>0$ we have the following inequality:*

$$\begin{aligned}
&\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \\
\leq\ & \frac{\eta(b,a)}{8}\left[\sup\left\{\left|f'(a)\right|,\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|\right\}+\sup\left\{\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|,\left|f'(a+\eta(b,a))\right|\right\}\right]. \qquad (14)
\end{aligned}$$

Note that, the inequality (14) coincides with the inequality (3) in Theorem 2.8.

Remark 3.7. *The inequality (13) is better than the inequality (5). Proof can be made similar to Remark 3.2.*

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References

[1] M. Alomari, M. Darus and U. S. Kirmaci, Refinements of Hadamard type inequalities for quasi-convex functions with applications to trapezoidal formula and to special means, Computers and Mathematics with Applications, 59 (2010) 225–232.
[2] A. Barani, A. G. Ghazanfari and S. S. Dragomir, Hermite-Hadamard inequality through prequasiinvex functions, RGMIA Research Report Collection, 14 (2011), Article 48, 7 pp.
[3] A. Barani, A. G. Ghazanfari and S. S. Dragomir, Hermite-Hadamard inequality for functions whose derivatives absolute values are preinvex, RGMIA Research Report Collection, 14 (2011), Article 64, 11 pp.
[4] A. Ben-Israel and B. Mond, What is invexity?, J. Austral. Math. Soc., Ser. B, 28(1) (1986) 1–9.
[5] S. S. Dragomir and R. P. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and trapezoidal formula, Appl. Math. Lett., 11(5) (1998) 91–95.
[6] M. A. Hanson, On sufficiency of the Kuhn-Tucker conditions, J. Math. Anal. Appl., 1 (1981) 545–550.
[7] D. A. Ion, Some estimates on the Hermite-Hadamard inequality through quasi-convex functions, Annals of University of Craiova, Math. Comp. Sci. Ser., 34 (2007) 82–87.
[8] İ. İşcan, New refinements for integral and sum forms of Hölder inequality, Preprint, arXiv:1901.05841v1 [math.GM], 8 Jan 2019.
[9] İ. İşcan, M. Kadakal and H. Kadakal, On two times differentiable preinvex and prequasiinvex functions, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat., 68(1) (2019) 950–963.
[10] M. Kadakal, İ. İşcan, H. Kadakal and K. Bekar, On improvements of some integral inequalities, Preprint, Researchgate, DOI:10.13140/RG.2.2.15052.46724, January 2019.
[11] H. Kadakal, M. Kadakal and İ. İşcan, New Type Integral Inequalities for Three Times Differentiable Preinvex and Prequasiinvex Functions, Open J. Math. Anal., 2(1) (2018) 33–46.
[12] M. A. Latif, Some inequalities for differentiable prequasiinvex functions with applications, Konuralp J. Math., 1(2) (2013) 17–29.
[13] D. S. Mitrinovic, J. E. Pečarić and A. M. Fink, Classical and new inequalities in analysis, Kluwer Academic Publishers, Dordrecth, Boston, London, 1993.
[14] M. A. Noor, Hermite-Hadamard integral inequalities for log-preinvex functions, J. Math. Anal. Approx. Theory, 2 (2007) 126–131.
[15] C. E. M. Pearce and J. Pečarić, Inequalities for differentiable mappings with application to special means and quadrature formulae, Appl. Math. Lett., 13(2) (2000) 51–55.
[16] R. Pini, Invexity and generalized convexity, Optimization, 22 (1991) 513–523.
[17] M. Z. Sarikaya, H. Bozkurt and N. Alp, On Hermite-Hadamard type integral inequalities for preinvex and log-preinvex functions, Contemporary Anal. Appl. Math., 1(2) (2013) 237–252.
[18] T. Weir and B. Mond, Preinvex functions in multiple bijective optimization, J. Math. Anal. Appl., 136 (1998) 29–38.
[19] S. -H. Wu, On the weighted generalization of the Hermite-Hadamard inequality and its applications, The Rocky Mountain J. of Math., 39(5) (2009) 1741–1749.
[20] X. M. Yang and D. Li, On properties of preinvex functions, J. Math. Anal. Appl., 256 (2001) 229–241.