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Practical stability analysis of perturbed fuzzy control system related to unperturbed fuzzy control system

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Abstract: The behavior of solutions of a perturbed fuzzy control system with respect to an original unperturbed fuzzy control system is investigated. Notions of practical stability, asymptotic stability and instability are introduced. Sufficient conditions of stability properties are given with the help of Lyapunov-like functions.

Key words: Fuzzy control differantial equation, practical stability, perturbed system, Lyapunov-like functions, comparison results

1. Introduction

Lyapunov's second method is a standard technique that allows estimation of the qualitative behavior of differential equations and stability analysis when the behavior of the apparent solution of the comparison system is known, without solving the system in nonlinear systems. This method indicates that the system is stable if the appropriate Lyapunov function is found. In this sense, this method is sufficient, because even if the Lyapunov function cannot be found, a suitable candidate showing the stability of the system can still be determined.

In some status, a system in theory may be stable or asymptotically stable, but it is unstable in practice essentially because the stable domain or the domain of attraction is not large enough to authorize the desired deflection to cancel out [26]. Conversely, occasionally the desired state of a system may be mathematically unstable and nevertheless the system may pendulate enough near this state in which its performance is admissible. In many problems of practical significance one is not only interested in the qualitative data provided by Lyapunov stability results, but also in quantitative data concerning the system's attitude such as prediction of trajectory bounds. In the light of all this information, practical stability concepts [12] are more influential. Accordingly, the research of fuzzy differential systems [6, 7],[2, 10, 13, 14, 24] is initiated and sufficient conditions, in terms of Lyapunov-like functions [15, 16] are provided for the practical stability, which consolidate Lyapunov's second method.

Fuzzy differential systems are robust tools for modeling uncertainty and for processing uncertain or nominative data in mathematical models, they have been applied to a large diversity of real problems, for example, quantum optics, gravity [3], population models, engineering applications [4] and some other models.

The concept of the fuzzy derivative was first introduced by Chang and Zadeh [1, 30]. The use of fuzzy

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differential equations is a natural way to model dynamic systems under possibilistic uncertainty. The concept of differential equations in a fuzzy structure was built by Kaleva [6, 7] and several authors have produced a large diversity of results in both the theoretical and applied fields [4, 11, 20, 28]. A diversity of exact, approximate, and entirely numerical methods are existing to discover the solution of a fuzzy initial value problem.

It was put forth that the fuzzy differential equations has specific disadvantages because the solution owns the feature that the diameter is nondecreasing as time increases. It was put forth that this course of the solutions is due to the fuzzification of the derivative used in the formulation of the fuzzy differential equations. As a result alternative formulations have been suggested.

The mathematical theory of control commence from that is interested in the basis underlying the analysis and design of control systems [18, 19]. They have been applied to a large diversity of real problems, such as in electronics [4], climate modeling, machine design [4] and neural networks [23] The fundamental problems that emerge for fuzzy control problems are: the existence and uniqueness of the solution involving fuzzy control, the accessibility and controllability of fuzzy control systems[21, 22].

It is studied about the stability of the fuzzy control differential equation in paper [5, 10, 20], about practical stability of the fuzzy differential equation in paper [8, 26, 28], about stability of perturbed system related to unperturbed system in paper [25, 27]. In the light of all these studies, Lyapunov's second method is applied on perturbed fuzzy control system related to unperturbed fuzzy control system and its practical stability was studied along with other stability features in this paper. With this approach, we aimed to expand the family of practical stable perturbed system because while the perturb system is not stable, it can be stable relative to the unperturbed system.

The paper is organized as follows: In Section 1, we present definitions and necessary background material. In Section 2, we introduce perturbed and unperturbed fuzzy control system, explain stability definitions and main comparison theorem. In Section 3, we present stability properties of fuzzy control differential equation by comparison differential equation. In Section 4, we clarify practical stability properties of perturbed fuzzy control differential equation with respect to the unperturbed fuzzy control differential equation. In Section 5, we have a comparison result in which the practical stability properties of null solution of the comparison system imply the corresponding practical stability properties of perturbed fuzzy control differential equation with respect to the unperturbed fuzzy control differential equation.

2. Preliminaries

In this section, we give the basic definitions of fuzzy algebra [10]. Let $K_c(R^n)$ be the collection of all nonempty compact, convex subsets of R^n. If $\alpha, \beta \in \mathbb{R}$ and $A, B \in K_c(R^n)$, then

$$\alpha(A+B) = \alpha A + \alpha B, \alpha(\beta A) = (\alpha\beta)A, 1A = A. \tag{2.1}$$

If $\alpha, \beta \geq 0$, then we have the equality $(\alpha+\beta)A = \alpha A + \beta A$.

Define the Hausdorff metric as follows:

$$D\left[A, B\right] = \max\left[\sup_{x\in B} d(x, A), \sup_{y\in A} d(y, B)\right] \tag{2.2}$$

where $d(x, A) = \inf\left[d(x, y) : y \in A\right]$, and A, B are bounded sets in R^n. The metric space $(K_c(R^n), D)$ is a complete metric space.

Let $D[A,B]$ be the Hausdorff metric between the sets $A,B,C \in K_c(R^n)$. Then we define the following properties:

$$D[A+C,B+C] = D[A,B] \tag{2.3}$$

$$D[A,B] = D[B,A] \tag{2.4}$$

$$D[\lambda A,\lambda B] = \lambda D[A,B] \tag{2.5}$$

$$D[A,B] \leq D[A,C] + D[C,B] \tag{2.6}$$

for all $A,B,C \in K_c(R^n)$ and $\lambda \in R_+$. It is known that $(K_c(R^n),D)$ is complete, seperable and locally compact.

Define $E^n = \{x : R^n \to [0,1]$ such that $x(t)$ satisfies (i)-(iv) stated below$\}$:

(i) x maps R^n onto $I=[0,1]$,

(ii) $[x]^a$ is compact subset of $\mathbb{R}^n$ for all $\alpha \in I$ where $a-$level set $x_a = [x]^a = \{z \in \mathbb{R}^n \mid x(z) \geq \alpha\}$,

(iii) $[x]^0$ is bounded subset of $\mathbb{R}^n$,

(iv) x is fuzzy convex, that is, for $0 \leq \lambda \leq 1$

$$x(\lambda z_1 + (1-\lambda)z_2) \geq \min\{x(z_1),x(z_2)\} \qquad \lambda \in [0,1]$$

We define the diameter of x as $diam[x]^a = \overline{x}(\alpha) - \underline{x}(a)$. Let us denote by $D_0[x_1,x_2] = \sup\{D([x_1]^a,[x_2]^a) : 0 \leq a \leq 1\}$ the distance between x_1 and x_2 in E^n, where $D([x_1]^a,[x_2]^a)$ is Hausdorff distance between two set $[x_1]^a,[x_2]^a$ of $K_c(R^n)$. Then (E^n,D_0) is complete space. Some properties of metric D_0 are as follows:

$$D_0[x_1+x_3,x_2+x_3] = D_0[x_1,x_2]\ , \tag{2.7}$$

$$D_0[\lambda x_1,\lambda x_2] = |\lambda|\, D_0[x_1,x_2]\ , \tag{2.8}$$

$$D_0[x_1,x_2] \leq D_0[x_1,x_3] + D_0[x_3,x_2] \tag{2.9}$$

for all $x_1,x_2,x_3 \in E^n$ and $\lambda \in \mathbb{R}$. Let $x_1,x_2 \in E^n$, if there exists $x_3 \in E^n$ such that $x_1 = x_2 + x_3$, then x_3 is called the H-difference of x_1,x_2 and it is denoted by $x_1 - x_2$. Let us remark that $x_1 - x_2 \neq x_1 + (-1)x_2$. Let us denote $\theta^n \in E^n$ the zero element of E^n as follows: $\theta^n(z)=1$ if $z=0$ and $\theta^n(z)=0$ if $z \neq 0$, where 0 is the zero element of R^n.

We define the space of continuous fuzzy functions as

$$C([t_0,T],E^n) = \{x : [t_0,T] \to E^n \mid x \text{ is continuous}\}$$

which is complete metric space endowed with the following metric

$$D_0^*[x_1,x_2] = \sup_{t\in[t_0,T]} D_0[x_1(t),x_2(t)] \ \text{ for } x_1,x_2 \in C([t_0,T],E^n).$$

In the following we recall some main concepts and properties of fuzzy Hukuhara differentiability for fuzzy functions.

Definition 2.1 [10] *Let $x : T \to E^n$ and $T \subset \mathbb{R}$ be compact interval. We say that x is differentiable at $t \in T$, if there exists a $Hukuhara-$ derivative $D_H x(t) \in E^n$, such that for all $h > 0$ sufficiently small, $Hukuhara-$ difference $x(t+h) - x(t)$, $x(t)-$ $x(t+h)$ exist and limits*

$$D_H x = \lim_{h \longrightarrow 0^+} \frac{x(t+h) - x(t)}{h} = \lim_{h \longrightarrow 0^+} \frac{x(t) - x(t-h)}{h} \tag{2.10}$$

provided that there limits are exists.The limit by metric D_0, $\lim_{h \longrightarrow 0^+} D_0[\frac{x(t+h)-x(t)}{h}, D_H x(t)] = 0$.

If x, y differentiable at t then $D_H(x+y)(t) = D_H x(t) + D_H y(t)$ and $D_H(\lambda x)(t)) = \lambda D_H x(t), \lambda \in R$. If $x : T \to E^n$ is differentiable then it is continuous.

Definition 2.2 [10] *We say that a mapping $x : T \to E^n$ is strongly measurable if for all $a \in [0,1]$ the set-valued mapping $x_a : T \to P_K(\mathbb{R}^n)$ defined by $x_a(t) = [x(t)]^a$ is (Lebesgue) measurable, when $P_K(\mathbb{R}^n)$ is endowed with the topology generated by Hausdorff metric D_0.*

If x is strongly measurable, then it is measurable with respect to the topology generated by d that is defined by

$$u \in E^n, \quad \{t \mid d(x(t), u) \leq \varepsilon\} = \underset{a\in[0,1]}{\cap} \{t \mid D_0(x_a(t), [u]^a) \leq \varepsilon\}$$

Definition 2.3 [10] *Let $x : T \to E^n$, Hukuhara integral of x over T, denoted $\int_T x(t)dt$ is defined levelwise by the equation*

$$\left[\int_T x(t)dt\right] = \int_T x_a(t)dt = \left\{\int_T \widehat{x}(t)dt \mid \widehat{x} : T \to \mathbb{R}^n \textit{ is a measurable selection for } x_a.\right\} \textit{ for all } a \in [0,1]$$

If $x : T \to E^n$ is strongly measurable and integrably bounded, then x is integrable, if it is continuous, then is also integrable and for all $t \in T$ the integral $G(t) = \int_{[a,t]} x$ is differentiable and $D_H G(t) = x(t)$.

If $x : T \to E^n$ is differentiable and assume that the derivative $D_H x$ is integrable over T. Then for each $s \in T$, we have $x(s) = x(a) + \int_{(a,s]} D_H x dt$.

If $x, y : T \to E^n$ are integrable, then the following properties of the integral are valid.

$$D_0[x(\cdot), y(\cdot)] : T \longrightarrow \mathbb{R} \text{ is integrable;} \tag{2.11}$$

$$D_0\left[\int_T x(t)\,dt, \int_T y(t)\,dt\right] \leq \int_T D_0[x(t), y(t)]\ dt; \tag{2.12}$$

$$\int_a^b x(s)\,dt = \int_a^c x(s)\,ds + \int_c^b x(s)\,ds \text{ for } a \leq c \leq b \tag{2.13}$$

By using the Housdorff metric, it follows that for $A \in E^n$

$$D_0[A, \theta] = \|A\| = \sup_{a\in A} \|a\|$$

where θ is the zero element of R^n which is a one-point set.

3. Perturbed fuzzy control system related to unperturbed fuzzy control system

In fuzzy metric space E^n, let us consider the inital valued problems (IVP) of nonlinear fuzzy control differential equations (NLFCDE's) in $[t_0, T]$;

$$D_H x(t) = f(t, x(t), u(t)),\ x(t_0) = x_0 \in E^n,\ u(t_0) = u_0 \in E^p \text{ and } t \in [t_0, T]\ \ t_0 \geq 0 \tag{3.1}$$

the perturbed system of (3.1)

$$D_H y(t) = F(t, y(t), u(t)),\ y(t_0) = y_0 \in E^n,\ u(t_0) = u_0 \in E^p \text{ and } t \in [t_0, T]\ \ t_0 \geq 0 \tag{3.2}$$

where $f, F : [t_0, T] \times E^n \times E^p \to E^n$ and admissible control $u(t) \in E^p$. We have a special case of (3.2) that is perturbation equation of (3.1) if $F(t, y(t), u(t)) = f(t, y(t), u(t)) + R(t, y(t), u(t))$ where $R(t, y(t), u(t))$ is the perturbation term. The above assumptions imply the existence of trivial solutions of (3.1) and (3.2) through (t_0, x_0, u_0) and (t_0, y_0, u_0), respectively.

Thus the corresponding with the IVP for NLFCDEs of (3.1), (3.2) are the followings respectively.

$$x(t) = x_0 + \int_{t_0}^{T} f(s, x(s), u(s))ds,\ t \in [t_0, T]\ ,\ x_0 \in E^n, u_0 \in E^p \tag{3.3}$$

$$y(t) = y_0 + \int_{t_0}^{T} F(s, y(s), u(s)))ds,\ t \in [t_0, T]\ ,\ x_0 \in E^n, u_0 \in E^p \tag{3.4}$$

Definition 3.1 *[20] Let $u(t) \in E^p$ be an admissible fuzzy control, which means at moment t_0, we have $x(t_0) = x(t_0, x_0, t_0, u(t_0)) = x_0 \in E^n$, for any $\overline{x} \in E^n$ exists $t_1 > t_0$ such that $x(t_1) = x(t_0, x_0, t_1, u(t_1)) = \overline{x}$ and the pair of fuzzy states $(x_0, \overline{x}) \in E^n$ is called controllable by $u(t)$.*

Before we can establish our comparison theorem and Lyapunov stability criteria we need to introduce the following definitions.

3.1. Stability criteria

We assume that NLFCDE in Equation (3.1) has the trivial solution, which means $f(t, \theta^n, u(t)) = \theta^n$.

Definition 3.1.1 *[10] The trivial NLFCDE solution $x(t) = x(t, t_0, x_0, u(t))$ of Equation (3.1) through (t_0, x_0) is said to be:*

(S) stable by Lyapunov's mean if: for each $\varepsilon > 0$ and $t_0 > 0$ there exists a $\delta = \delta(t_0, \varepsilon)$ such that $D_0[x_0, \theta^n] < \delta$ implies $D_0[x(t), \theta^n] < \varepsilon$ for $t \geq t_0$

(US) uniformly stable by Lyapunov's mean if: if δ in (S) is independent of $t_0 \in \mathbb{R}_+$.

(AS) asymptotically stable by Lyapunov's mean if: it is stable and $\lim_{t\to\infty} D_0[x(t), \theta^n] = 0$

(ES) exponentially stable by Lyapunov's mean if: $D_0[x(t), \theta^n] \leq \beta(D_0[x_0, \theta^n], t_0) \exp[-a(t - t_0)]$, $t \geq t_0$ where $\beta(D_0[.,.], t_0) : [0, 1] \times \mathbb{R}_+ \to \mathbb{R}_+$ and $a > 0$

(PS) *practical stable by Lyapunov's mean if: given* (λ, A) *with* $0<\lambda<A$ *there exists a* $(\lambda, A)\geq 0$ *such that* $D_0[x_0,\theta^n]<\lambda$ *implies* $D_0[x(t),\theta^n]<A$, $t\geq t_0$ *for some* $t_0\in\mathbb{R}_+$.

(UPS) *uniformly practical stable by Lyapunov's mean if:* (PS) *holds for every* $t_0\in\mathbb{R}_+$.

(PQS) *practically quasi stable by Lyapunov's mean if: given* $(\lambda, B, T)>0$, $0<\lambda<B$ *and* $t_0\in\mathbb{R}_+$ *such that* $D_0[x_0,\theta^n]<\lambda$ *implies* $D_0[x(t),\theta^n]<B$, $t\geq t_0+T$.

Definition 3.1.2 *[25] We assume that NLFCDE the solution* $y(t)=y(t,t_0,y_0,u(t))$ *of system (3.2) through* (t_0,y_0) *for* $t\geq t_0$

(SW) *stable with respect to the solution* $x(t)=\ x(t,t_0,x_0,u(t))$ *where* $x(t,t_0,x_0,u(t))$ *any solution of system (3.1) for* $t\geq t_0$ *if and only if given any* $\varepsilon>0$ *and* $t_0>0$ *there exist a* $\delta=\delta(t_0,\varepsilon)$ *such that* $D_0[y_0-x_0,\theta^n]<\delta$ *implies* $D_0[y(t)-x(t),\theta^n]<\varepsilon$ *for* $t\geq t_0$,

(USW) *uniformly stable with respect to the solution* $x(t)=\ x(t,t_0,x_0,u(t))$ *if* δ *in* (SW) *is independent of* $t_0\in\mathbb{R}_+$.

(ASW) *asymptotically stable with respect to the solution* $x(t,t_0,x_0,u(t))$ *if* (SW) *holds and there exists* $\gamma(t_0)>0$ *such that* $\lim_{t\to\infty} D_0[y(t)-x(t),\theta^n]=0$ *with*

$D_0[y_0-x_0,\theta^n]<\ \gamma(t_0)$

(ESW) *exponentially stable with respect to the solution* $x(t,t_0,x_0,u(t))$ *if there exists an estimate such that* $D_0[y(t)-x(t),\theta^n]\leq D_0[y_0-x_0,\theta^n]\exp[-a(t-t_0)]$, *for all* $a>0$

(PSW) *practical stable with respect to the solution* $x(t,t_0,x_0,u(t))$ *if given* (λ, A) *with* $0<\lambda<A$ *there exists a* $(\lambda, A)\geq 0$ *such that* $D_0[y_0-x_0,\theta^n]<\lambda$ *implies* $D_0[y(t)-x(t),\theta^n]<A$, $t\geq t_0$ *for some* $t_0\in\mathbb{R}_+$.

$(UPSW)$ *uniformly practical stable with respect to the solution* $x(t,t_0,x_0,u(t))$ *if* (PSW) *holds for every* $t_0\in\mathbb{R}_+$.

$(PQSW)$ *practically quasi stable with respect to the solution* $x(t,t_0,x_0,u(t))$ *if given* $(\lambda, B, T)>0$ *with* $0<\lambda<B$ *and* $t_0\in\mathbb{R}_+$ *such that* $D_0[y_0-x_0,\theta^n]<\lambda$ *implies* $D_0[y(t)-x(t),\theta^n]<B$, $t\geq t_0+T$ *for some* $t_0\in\mathbb{R}_+$.

Definition 3.1.3 *[10] A function* $\varphi(r)$ *is said to belong to the class* K *if* $\varphi\in C[(0,\rho),\mathbb{R}_+]$, $\varphi(0)=0$, *and* $\varphi(r)$ *is strictly monotone increasing in* r. It is said to belong to class κ_∞ if $\rho=\infty$ and $\varphi(r)\to\infty$ as $r\to\infty$.

Definition 3.1.4 *[10]* For a real-valued function $V(t,x(t),u(t))\in C[\mathbb{R}_+\times E^n\times E^p,E^n]$ we define the Dini derivatives as follows:

$$D^+V(t,x,u)\equiv\lim_{h\to 0^+}\sup\frac{1}{h}[V(t+h,x+hf(t,x,u),u(t))-V(t,x,u)]$$

$$D_-V(t,x,u)\equiv\lim_{h\to 0^-}\inf\frac{1}{h}[V(t+h,x+hf(t,x,u),u(t))-V(t,x,u)]$$

for $(t,x,u)\in\mathbb{R}_+\times E^n\times E^p$.

Definition 3.1.5 *[25] For a real-valued function* $V\in C[\mathbb{R}_+\times E^n\times E^p,E^n]$ *we define the generalized derivatives (Dini-like derivatives) as follows:*

$$D^+_*V(t,y-x,u)=\lim_{h\to 0^+}\sup\frac{1}{h}[V(t+h,y-x+h[F(t,x,u)-f(t,x,u)],u(t))-V(t,y-x,u)]$$

$$D_*^- V(t, y-x, u) = \lim_{h \to 0^-} \inf \frac{1}{h}[V(t+h, y-x+h[F(t,x,u) - f(t,x,u)], u(t)) - V(t, y-x, u)]$$

for $(t, y-x, u) \in \mathbb{R}_+ \times E^n \times E^p$ *where* $y(t) = y(t, t_0, y_0, u(t))$ *is the solution of the system (3.2) and* $x(t) = x(t, t_0, x_0, u(t))$ *is any solution of the system (3.1) for* $t \geq t_0$ *and some* $t_0 \in \mathbb{R}_+$*.*

3.2. Comparison theorem

We consider comparison system to predict the stability properties of $y(t, t_0, y_0, u(t))$ solution of (3.2) with respect to $x(t, t_0, x_0, u(t))$ any solution of the system (3.1).

Teorem 3.2.1 *Assume that* f *,* F $: [t_0, T] \times E^n \times E^p \to E^n$ *and*

i)

$$\lim_{h \to 0^+} \sup \frac{1}{h}[D_0[y - x + h(F(t,y,u) - f(t,x,u)), \theta^n] - D_0[y-x, \theta^n]] \leq G(t, D_0[y-x, \theta^n])$$

where $G \in C[[t_0, T] \times \mathbb{R}_+, \mathbb{R}]$*;*

ii) $r(t) = r(t, t_0, z_0)$ *is maximal solution of the scalar differential equation exists on* $[t_0, T]$*,*

$$z' = G(t, z), \; z(t_0) = z_0 \geq 0 \text{ for } t \geq t_0 \tag{3.5}$$

Then if $x(t)$ *ve* $y(t)$ *are solution of (3.1) and (3.2) through* (t_0, x_0) *and* (t_0, y_0) *respectively on* $[t_0, T]$*.*

We have

$$D_0[y(t, t_0, y_0, u(t)) - x(t, t_0, x_0, u(t)), \theta^n] \leq r(t, t_0, z_0) \textit{ provided that } D_0[y_0 - x_0, \theta^n] \leq r_0$$

Proof For small $h > 0$ the H-difference of $x(t+h) - x(t)$, $y(t+h) - y(t)$ exist. Define $m(t) = D_0[y-x, \theta^n]$ and we have for $t \in [t_0, T]$

$$m(t+h) - m(t) = D_0[y(t+h) - x(t+h), \theta^n] - D_0[y(t) - x(t), \theta^n]$$

By using the triangular inequality for D_0, we obtain

$$D_0[y(t+h) - x(t+h), \theta^n] \leq D_0[y(t+h) - x(t+h), \; y(t) - x(t) + h(F(t, y(t), u(t)) - f(t, x(t), u(t)))] \\ + D_0[y(t) - x(t) + h(F(t, y(t), u(t)) - f(t, x(t), u(t))), \theta^n]$$

Hence, it follows that

$$\frac{m(t+h) - m(t)}{h} \leq \frac{1}{h} D_0[y(t+h) - x(t+h) - y(t) - x(t), h(F(t, y(t), u(t)) - f(t, x(t), u(t)))] \\ + \frac{1}{h}[D_0[y(t) - x(t) + h(F(t, y(t), u(t)) - f(t, x(t), u(t))), \theta^n] - D_0[y(t) - x(t), \theta^n]]$$

since the properties of D_0 and the fact that $x(t)$ and $y(t)$ are the solutions of (3.1) ve (3.2) respectively, we get

$$D^+m(t) = \limsup_{h\to 0^+}\frac{m(t+h)-m(t)}{h} \leq$$

$$\limsup_{h\to 0^+} D_0[\frac{y(t+h)-x(t+h)-[y(t)-x(t)]}{h}, [F(t,y(t),u(t))-f(t,x(t),u(t))]]$$

$$+\limsup_{h\to 0^+}\frac{1}{h}[D_0[y(t)-x(t)+h(F(t,y(t),u(t))-f(t,x(t),u(t))),\theta^n]-D_0[y(t)-x(t),\theta^n]]$$

This implies that

$$D^+m(t) \leq G(t, D_0[y-x,\theta^n]) = G(t,m(t))$$

and by using the comparison result in Theorem 1.4.1 given in Lakshmikantham and Leela **[5].**Therefore, we have

$$m(t) = D_0[y(t,t_0,y_0,u(t)) - x(t,t_0,x_0,u(t)),\theta^n] \leq r(t,t_0,z_0) \text{ provided that } D_0[y_0-x_0,\theta^n] \leq z_0$$

Corollory 3.2.2 *The function* $G(t,z) = 0$ *is admissible in Theorem 3.2.1 to yield the estimate* $D_0[y(t,t_0,y_0,u(t)) - x(t,t_0,x_0,u(t)),\theta^n] \leq D_0[y_0-x_0,\theta^n]$.

4. Stability criteria of fuzzy control differential equations

We assume that NLFCDE in Equation (3.1) has the trivial solution, which means $f(t,\theta^n,u(t)) = \theta^n$.

Teorem 4.1 *Assume that* $f : [t_0,T]\times S_\rho \times E^p \to E^n$ *and where* $S_\rho = [y-x \in E^n : D_0[y-x,\theta^n] < \rho]$

i)

$$\lim_{h\to 0^+}\sup\frac{1}{h}[D_0[x+hf(t,x,u),\theta^n] - D_0[x,\theta^n]] \leq G(t, D_0[x,\theta^n]) \tag{4.1}$$

ii) Let $r(t) = r(t,t_0,z_0)$ *be the maximal solution of the scalar differential equation*

$$z' = G(t,z),\ z(t_0) = z_0 \geq 0 \text{ for } t \geq t_0 \text{ and } G(t,z) = 0 \tag{4.2}$$

Then the stability properties of trivial solution of scalar differential equation imply the corresponding stability properties of trivial solution of fuzzy differential equation (3.1) respectively.

Proof Let the trivial solution of (4.2) be stable. Then, given $\epsilon > 0$ and $t_0 \in \mathbb{R}_+$, there exists a positive $\delta = \delta(t_0,\epsilon)$ with the property $0 \leq w_0 \leq \delta$ implies $w(t,t_0,w_0) < \epsilon,\ t \geq t_0$ where $w(t,t_0,w_0)$ is any solution of scalar differential equation. We claim that with these ϵ,δ the trivial solution $x(t) = 0$ of is stable. If this is false, there would exist a solution $x(t) = x(t,t_0,x_0,u(t))$ of (3.1) with $D_0[x_0,\theta^n] < \delta$ and $t_1 > t_0$ such that $D_0[x_1,\theta^n] = \epsilon$ and $D_0[x,\theta^n] \leq \epsilon < \rho,\ \ t_0 \leq t \leq t_1$. For $[t_0,t_1]$ using condition (3.1), corollary 3.4.1- **(**Lakshmikantham, V. and Mohapatra R.N., 2003, Theory of Fuzzy Differential Equations and Inclusions **[3].)** yields the estimate $D_0[x,\theta^n] \leq r(t,t_0,D_0[x_0,\theta^n]) < \epsilon$ proving the claim.

5. Practical stability of perturbed fuzzy control system related to unperturbed fuzzy control system

Teorem 5.1 *Assume that the following hold* *i) Let Lyapunov-like function* $V(t,x(t),u(t)) \in C(R_+ \times E^n \times E^p, E^n)$, $| V(t,x,u_1) - V(t,y,u_2) | \leq L(D_0[x,y] + D_0[u_1,u_2])$ $L > 0$ *bounded Lipschitz constant and for* $(t,x,u) \in R_+ \times S_\rho \times E^p$ *where* $S_\rho = [y - x \in E^n : D_0[y-x,\theta^n] < \rho]$ *such that*

$$D^+V(t,y-x,u) = \lim_{h \to 0^+} \sup \frac{1}{h}[V(t+h, y-x+h(F(t,y,u)-f(t,x,u)), u(t)) - V(t,y-x,u)] \leq 0 \quad (5.1)$$

ii)Let $V(t,y(t)-x(t),u(t)) \in C(R_+ \times E^n \times E^p, E^n)$ *and* $a,b \in \kappa$,

$$b(D_0[y(t,y,u(t)) - x(t,x,u(t)),\theta^n]) \leq V(t,y(t)-x(t),u(t)) \leq a(t, D_0[y(t,y,u(t)) - x(t,x,u(t)),\theta^n]) \quad (5.2)$$

Then $y(t,t_0,y_0,u(t))$ *the solution of FCDE (3.2) with respect to* $x(t,t_0,x_0,u(t))$ *any solution of FCDE (3.1) is practical stable for (3.1) for* $t \geq t_0$.

Proof Let us assume that given (λ, A) with $0 < \lambda < A$. Then it is possible to find a $b > 0$ such that $a(t,\lambda) < b(A)$. Practical stability holds such that

$$D_0[y(t,t_0,y_0,u(t)) - x(t,t_0,x_0,u(t)),\theta^n] < A \text{ provided that } D_0[y_0 - x_0] < \lambda \quad (5.3)$$

If the fuzzy control differential equation (3.1) is not practically stable and then there would exist a solution of fuzzy control differential equation $y(t,t_0,y_0,u(t))$ the solution of (3.2) with respect to $x(t,t_0,x_0,u(t))$ is any solution of fuzzy control differential equation (3.1) for $t \geq t_0$ and exist $t_1 > t_0$ with $D_0[x_0 - y_0,\theta^n] < \lambda$ satisfiying

$$D_0[y(t_1,t_0,y_0,u(t)) - x(t_1,t_0,x_0,u(t)),\theta^n] = A \quad t \in [t_0,t_1]$$

So that we have, because of (5.1) and (5.2) $b(A) \leq V(t_1, y(t_1,t_0,y_0,u(t)) - x(t_1,t_0,x_0,u(t)), u(t))$ for $t_1 > t_0$. This means that $D_0[y(t_1,t_0,y_0,u(t)) - x(t_1,t_0,x_0,u(t)),\theta^n] < \rho$ for $t \in [t_0,t_1]$ and hence we get from the assumptions (5.1) and Corollory 3.2.2, the estimate

$$V(t_1, y(t_1,t_0,y_0,u(t)) - x(t_1,t_0,x_0,u(t)), u(t)) \leq V(t_0, y_0 - x_0, u(t)) \quad t \geq t_0$$

We get

$$\begin{aligned} b(A) &= b(D_0[y(t_1,t_0,y_0,u(t)) - x(t_1,t_0,x_0,u(t)),\theta^n]) \\ &\leq V(t_1, y(t_1,t_0,y_0,u(t)) - x(t_1,t_0,x_0,u(t)), u(t)) \\ &\leq V(t_0, y_0 - x_0, u(t)) \leq a(t_0, D_0[y_0 - x_0,\theta^n]) \leq a(t_0,\lambda) < b(A) \end{aligned}$$

which contradicts. Hence (5.3) is valid and we have $y(t,t_0,y_0,u(t))$ the solution of (3.2) with respect to $x(t,t_0,x_0,u(t))$ is practical stable. Since λ is now dependent of t_0, we have uniformly practically stability $y(t,t_0,y_0,u(t))$ the solution of (3.2) with respect to $x(t,t_0,x_0,u(t))$.

Teorem 5.2 *Assume that following hold*

i) *Let* $V(t,x(t),u(t)) \in C(R_+ \times E^n \times E^p, E^n)$ $| V(t,x,u_1) - V(t,y,u_2) | \leq L(D_0[x,y] + D_0[u_1,u_2])$ $L > 0$ *and for* $(t,x,u) \in R_+ \times S_\rho \times E^p$ *and where* $S_\rho = [y - x \in E^n : D_0[y-x,\theta^n] < \rho]$ *such that*

$$D^+V(t, y-x, u) = \lim_{h\to 0^+} \sup \frac{1}{h}[V(t+h, y-x+h(F(t,y,u)-f(t,x,u)), u(t)) - V(t, y-x, u)] \\ \leq -\mu V(t_0, y_0 - x_0, u(t)) \tag{5.4}$$

ii) *Let* $V(t, y(t)-x(t), u(t)) \in C(R_+ \times S_\rho \times E^p, E^n)$

$$b(D_0[y(t)-x(t), \theta^n]) \leq V(t, y(t)-x(t), u(t))) \leq a(t, D_0[y(t)-x(t), \theta^n]) \quad a, b \in \kappa \tag{5.5}$$

Then $y(t,t_0,y_0,u(t))$ *the solution of FCDE (3.2) with respect to* $x(t,t_0,x_0,u(t))$ *any solution of FCDE (3.1) is practically quasi stable for* $t \geq t_0$.

Proof It is clear from (5.4) that we have practical stability of $y(t,t_0,y_0,u(t))$ the solution of (3.2) with respect to $x(t,t_0,x_0,u(t))$ any solution of the system (3.1) for $t \geq t_0$. Hence taking $B = \rho$ and designating $\lambda_0 = \lambda_0(t_0, \rho) = \lambda > 0$, we have by Theorem 1.We have practical stabillity with this λ_0, practical stability holds such that

$$D_0[y(t,t_0,y_0,u(t)) - x(t,t_0,x_0,u(t)), \theta^n] < \rho \text{ provided that } D_0[y_0 - x_0, \theta^n] < \lambda \text{ for } t \geq t_0 + T \tag{5.6}$$

If the theorem is false, then there would exist a solution of $y(t,t_0,y_0,u(t))$ the solution of (3.2) with respect to $x(t,t_0,x_0,u(t))$ for $t_1 > t_0 + T$ and following status be provided; with $D_0[y_0 - x_0, \theta^n] < \lambda$ satisfying

$$D_0[y(t_1,t_0,y_0,u(t)) - x(t_1,t_0,x_0,u(t)), \theta^n] = B \quad t \in [t_0 + T, t_1]$$

Consequently, we get from assumption (5.4), the estimate

$$V[t, y(t,t_0,y_0,u(t)) - x(t,t_0,x_0,u(t)), u(t)] < V(t_0, y_0 - x_0, u(t)) \exp[-\mu(t - t_0)], \quad t \geq t_0 + T \tag{5.7}$$

Given $B > 0$, we choose $T = T(t_0, B) = \frac{1}{\mu} In[\frac{a(t_0,\lambda_0)}{b(B)}] + 1$. Then we have from (5.4),(5.5),(5.6),(5.7), we get

$$\begin{aligned} b(D_0[y(t_1,t_0,y_0,u(t)) - x(t_1,t_0,x_0,u(t)), \theta^n]) &\leq V(t_1, y(t,t_0,y_0,u(t)) - x(t_1,t_0,x_0,u(t)), u(t)) \\ &\leq V(t_0, y_0 - x_0, u(t)) \exp[-\mu(t_1 - t_0)] \\ &\leq a(t_0, \lambda) \exp[-\mu(t_1 - t_0)] \\ &< b(A) \end{aligned}$$

This contradicton gives us practical stability of $y(t,t_0,y_0,u(t))$ the solution of (3.2) with respect to $x(t,t_0,x_0,u(t))$ any solution of the system (3.1) is quasi practically stable for $t \geq t_0$. Since λ is now dependent of t_0, we have uniformly practically stability $y(t,t_0,y_0,u(t))$ the solution of (3.2) with respect to $x(t,t_0,x_0,u(t))$.

6. A comparison result in practical stability of fuzzy control differential equations

In this section, we have a useful comprasion theorem in practical stability of fuzzy control differential systems via scalar differential equation and proof of this theorem.

Teorem 6.1 *Assume that*

i) *Let* $V(t, y(t)-x(t), u(t)) \in C(R_+ \times E^n \times E^p, E^n)$ $\quad | V(t,x,u_1) - V(t,y,u_2) \,|\leq L(D_0[x,y] + D_0[u_1,u_2])$ $\; L > 0$ *and for* $(t,x,u) \in R_+ \times S_\rho \times E^p$ *and*

$$b(D_0[y(t)-x(t),\theta^n]) \leq V(t, y(t)-x(t), u(t))) \leq a(t, D_0[y(t)-x(t),\theta^n]) \quad a, b \in \kappa \tag{6.1}$$

Dini derivatives of Lyapunov functions and comprasion of the scalar differential equation (6.2)

$$\begin{aligned} D^+V(t, x-y, u) &= \lim_{h\to 0^+} \sup \frac{1}{h}[V(t+h, y-x+h(F(t,y,u)-f(t,x,u)), u(t)] - V[t, y-x, u]] \\ &\leq g(t, V(t, y(t)-x(t)), \quad g(t,V) \in C[\mathbb{R}_+^2, \mathbb{R}] \end{aligned} \tag{6.2}$$

ii) *Let* $r(t) = r(t,t_0,z_0)$ *be the maximal solution of the scalar differential equation*

$$z' = g(t,z), \; z(t_0) = z_0 \geq 0 \text{ for } t \geq t_0 \tag{6.3}$$

Then the practical stability properties of the comparison differential equation imply the corresponding practical stability properties of $y(t,t_0,y_0,u(t))$ *the solution of fuzzy control differential equation (3.2) with respect to* $x(t,t_0,x_0,u(t))$ *any solution of the system (3.1) for* $t \geq t_0$.

Proof Suppose that comparison equation is practically stable, then for given any (λ, A) with $0 < \lambda < A$ and there exists $b = b(A)$ and $b \in \kappa$ such that

$$z(t,t_0,z_0) < b(A) \text{ provided that } d_s[z_0, 0] < \lambda, \; t \geq t_0 \tag{6.4}$$

We claim that with this λ, practical stability holds such that

$$D_0[y(t,t_0,y_0,u(t)) - x(t,t_0,x_0,u(t)), \theta^n] < A \text{ provided that } D_0[y_0 - x_0, \theta^n] < \lambda \text{ for } t \geq t_0 \tag{6.5}$$

If the theorem is false, then there would exist solution of fuzzy control differential equation; $y(t,t_0,y_0,u(t))$ the solution of fuzzy control differential equation (3.2) with respect to $x(t,t_0,x_0,u(t))$ any solution of the system (3.1) for $t \geq t_0$ exist a $t_1 > t_0$ and following status be provided; exist $D_0[y_0 - x_0, \theta^n] < \lambda$ for $t \geq t_0$ satisfying

$$D_0[y(t_1,t_0,y_0,u(t)) - x(t_1,t_0,x_0,u(t)), \theta^n] = A \tag{6.6}$$

for $t \in [t_0, t_1]$. Choose $z_0 = a(t_0, D_0[y_0 - x_0, \theta^n])$, we get the inequality

$$V(t, y(t)-x(t), u(t)) \leq r(t,t_0,z_0) \quad t \in [t_0, t_1] \tag{6.7}$$

So using (6.1) and (6.6), we have

$$b(A) \leq V(t_1, y(t_1,t_0,y_0,u(t)) - x(t_1,t_0,x_0,u(t)), u(t)) \quad t_1 > t_0 \tag{6.8}$$

This means that $D_0[y(t)-x(t),\theta^n] < \rho$ for $t \in [t_0,t_1]$ and hence we have the inequality

$$V(t_1, y(t_1,t_0,y_0,u(t)) - x(t_1,t_0,x_0,u(t)), u(t)) \leq r(t_1,t_0,z_0) \quad t \geq t_0 \tag{6.9}$$

By using (6.4) , (6.5), (6.6) and (6.7), we get

$$\begin{aligned} b(A) &= b(D_0[y(t_1,t_0,y_0,u(t)) - x(t_1,t_0,x_0,u(t)),\theta^n]) \\ &\le V(t_1, y(t_1,t_0,y_0,u(t)) - x(t_1,t_0,x_0,u(t)), u(t)) \\ &\le r(t_1,t_0,z_0) \\ &\le r(t_1,t_0,a(t_0,D_0[y_0 - x_0,\theta^n])) \\ &\le r(t_1,t_0,a(t_0,\lambda_1)) \\ &< b(A) \end{aligned}$$

This contradiction gives us practical stability properties of $y(t,t_0,y_0,u(t))$ the solution of fuzzy control differential equation (3.2) with respect to $x(t,t_0,x_0,u(t))$ any solution of the system (3.1) for $t \ge t_0$.

Assume that the comparison system is practically quasi-stable. Given any $b(B) > 0$, there exists a $T = T(t_0,B) > 0$ such that

$$0 < z_0 = a(t_0, d_s[z_0,0]) < \lambda \quad \text{implies} \quad z(t,t_0,z_0) < b(B),\ t \ge t_0 + T$$

Setting $z_0 = a(t,d_s[z_0,0]) < \lambda$ and using (i);

$$V(t_0, y_0 - x_0, u(t)) \le a(t_0, d_s[z_0,0]) = z_0 < \lambda$$

By using (i),

$$\begin{aligned} b(D_0[y(t,t_0,y_0,u(t)) - x(t,t_0,x_0,u(t)),\theta^n]) &\le V(t, y(t,t_0,y_0,u(t)) - x(t,t_0,x_0,u(t)), u(t)) \\ &\le r(t,t_0,z_0) \\ &< b(B) \text{ for } t \ge t_0 + T \end{aligned}$$

We obtain

$$D_0[y(t,t_0,y_0,u(t)) - x(t,t_0,x_0,u(t)),\theta^n] < B,\ t \ge t_0 + T$$

since $b \in \kappa$. This proves the practically quasi-stability.

7. Conclusion

While some systems are unstable, they may be stable compared to another system. We chose these two systems as perturb and unperturb systems. While the perturb system is not stable, it can be stable relative to the unperturbed system.

1- We improved some theorem for practical stability properties for this comparative system.

2- With this approach, we expand the family of practical stable perturbed system.

3- We developed a new comparison principle for nonlinear diferential systems, then we proved several practical stability criteria for fuzzy control system.

This study was the preliminary study for the inital time diference practical stability. We will see its usefulness by integrating the method analytically and numerically into the real problem.

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