A NOTE ON SOME CHARACTERIZATIONS OF CURVES DUE TO BISHOP FRAME IN EUCLIDEAN PLANE E^2

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Abstract

In this paper, we first obtain the differential equation characterizing position vector of a regular curve in Euclidean plane E^2. Then we study the special curves such as Smarandache curves, curves of constant breadth due to the Bishop frame in Euclidean plane E^2. We give some characterizations of these special curves due to the Bishop frame in Euclidean plane E^2.

AMS Subject Classification: 53A35, 53A40, 53B25

Key Words: A regular plane curve, Euclidean plane, Bishop frame, Smarandache curves, Curves of constant breadth.

E^2 ÖKLİD DÜZLEMİNDE BİSHOP ÇATISINA GÖRE EĞRİLERİN BAZI KARAKTERİZASYONLARI ÜZERİNE BİR İNCELEME

Özet

Bu makalede, öncelikle E^2 Öklid düzleminde regüler bir eğrinin konum vektörünü karakterize eden diferensiyel denklemi elde ediyoruz. Sonra Smarandache eğrileri, sabit genişlikli eğriler gibi özel eğrileri E^2 Öklid düzleminde Bishop çatısına göre inceliyoruz. Bu özel eğrilerin bazı karakterizasyonlarını veriyoruz.

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Anahtar kelimeler: Regüler düzlem eğrisi, Öklidyen düzlem, Bishop çatısı, Smarandache eğrileri, Sabit genişlikli eğriler.

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1. INTRODUCTION

There are lots of interesting and important problems in the theory of curves at differential geometry. One of the interesting problems is the problem of characterization of a regular curve in the theory of curves in the Euclidean and Minkowski spaces, see, [4], [9].

Special curves are classical differential geometric objects. These curves are obtained by assuming a special property on the original regular curve. Some of them are Smarandache curves, curves of constant breadth, Bertrand curves, and Mannheim curves, etc. Studying curves can be differed according to frame used for curve. Recently, in the studies of classical differential geometry of curves, one of the most used frames is parallel transport frame, also called Bishop frame which is an alternative frame needed for non-continously differentiable curves on which Bishop (parallel transport frame) frame is well defined and constructed in Euclidean and its ambient spaces [2].

Smarandache curves are regular curves whose position vectors are obtained by the Bishop frame vectors on another regular curve [16]. M. Turgut and S. Yılmaz have defined a special case of such curves and call it Smarandache TB_2 curves in the space E_1^4 [16] and Turgut also studied Smarandache breadth of pseudo null curves in E_1^4 [17]. A.T. Ali has introduced some special Smarandache curves in the Euclidean space [1]. Moreover, special Smarandache curves have been investigated by using Bishop frame in Euclidean space [3].

Curves of constant breadth were introduced by L.Euler [5]. M. Fujivara had obtained a problem to determine whether there exist space curve of constant breadth or not, and he defined breadth for space curves on a surface of constant breadth [6] .Some geometric properties of plane curves of constant breadth were given in [12]. And, in another work [13], these properties were studied in the Euclidean 3-space E^3. Moreover, In [15], these curves were studied in four dimensional Euclidean space E^4.

In the scope of our study we will take Smarandache curves, and curves of constant breadth into consideration due to the Bishop frame in Euclidean plane E^2. As much as we look at the classical differential geometry literature of the works in Euclidean plane E^2, the works were rare, see, [7], [8], [10], [11]. First, we obtain the differential equation characterizing position vector of curve due to the Bishop frame in Euclidean plane E^2. Then we study the special curves due to the Bishop frame in Euclidean plane E^2. We give some characterizations of these special curves in E^2.

2. PRELIMINARIES

Let E^2 be the Euclidean plane with metric

$$g = dx_1^2 + dx_2^2, \tag{1}$$

where x_1 and x_2 are rectangular coordinate system. A vector r of E^2 is said to be spacelike if $g(r,r) > 0$, or $r = 0$, timelike if $g(r,r) < 0$ and null if $g(r,r) = 0$ for $r \neq 0$ [7].

A curve x is a smooth mapping

$$x : I \to E^2,$$

from an open interval I onto E^2. Let s be an arbitrary parameter of x, then we denote the orthogonal coordinate representation of x as $x = (x_1(s), x_2(s))$ and also the vector

$$\frac{dx}{ds} = (\frac{dx_1}{ds}, \frac{dx_2}{ds}) = T \tag{2}$$

is called the tangent vector field of the curve $x = x(s)$ and also $\langle T, T \rangle = 1$ [7].

In the rest of the paper, we shall consider curves due to the Bishop frame. The Bishop derivative formula is given as follows:

$$\begin{bmatrix} T' \\ M_1' \end{bmatrix} = \begin{bmatrix} 0 & k_1 \\ -k_1 & 0 \end{bmatrix} . \begin{bmatrix} T \\ M_1 \end{bmatrix}, \tag{3}$$

where

$$k_1 = k_1(s) \tag{4}$$

is the curvature of the unit speed curve $x = x(s)$. The vector field M_1 is called the second Bishop vector field of the curve $x(s)$.

3. POSITION VECTOR OF A CURVE IN E^2

Let $\alpha = \alpha(s)$ be an unit speed curve due to the Bishop frame in the plane E^2. Then we can write position vector of $\alpha(s)$ with respect to the Bishop frame as

$$X = X(s) = \mu_1 T + \mu_2 M_1, \tag{5}$$

where μ_1 and μ_2 are arbitrary functions of s. Differentiating (5) and using Frenet equtaions we have a system of ordinary differential equations as follows:

$$\begin{cases} \dfrac{d\mu_1}{ds} - \mu_2 k_1 - 1 = 0, \\ \dfrac{d\mu_2}{ds} + \mu_1 k_1 = 0. \end{cases} \tag{6}$$

Using $(6)_1$ in $(6)_2$ we obtain

$$\frac{d}{ds}[\frac{1}{k_1}(\frac{d\mu_1}{ds} - 1)] + \mu_1 k_1 = 0. \tag{7}$$

The differential equation of second order, according to μ_1, is a characterization for the curve $x = x(s)$. Using change of variable $\theta = \int_0^s k_1 ds$ in (7), we arrive

$$\frac{d(\frac{1}{k_1})}{d\theta} = \frac{d^2\mu_1}{d\theta^2} + \mu_1. \tag{8}$$

By the method of variation of parameters and solution of (8) we have

$$\mu_1 = \cosh\theta \left[A - \int_0^\theta \frac{1}{k_1} \sinh\theta d\theta \right] + \sinh\theta \left[B + \int_0^\theta \frac{1}{k_1} \cosh\theta d\theta \right],$$

where $A, B \in R$. Rewriting the change of variable, we get

$$\begin{aligned} \mu_1 = {} & \cosh(\int_0^s k_1 ds) \left[A - \int_0^\theta \sinh(\int_0^s k_1 ds) d\theta \right] \\ & + \sinh(\int_0^s k_1 ds) \left[B + \int_0^\theta \cosh(\int_0^s k_1 ds) d\theta \right]. \end{aligned} \tag{9}$$

Denoting differentiation of the equation (9) as $\dfrac{d\mu_1}{ds} = l(s)$, and using (6) we have

$$\mu_2 = \frac{1}{k_1}\left[l(s)-1\right]. \tag{10}$$

Hence we give the following theorem:

Theorem 3.1. Let $\alpha = \alpha(s)$ be an arbitrary unit speed curve due to the Bishop frame in Euclidean plane E^2, Position vector of the curve $\alpha = \alpha(s)$ with respect to the Bishop frame can be composed by the following equation

$$\begin{aligned} X &= X(s) \\ &= (\cosh(\int_0^s k_1 ds)\left[A - \int_0^\theta \sinh(\int_0^s k_1 ds) d\theta\right] \\ &\quad + \sinh(\int_0^s k_1 ds)\left[B + \int_0^\theta \cosh(\int_0^s k_1 ds) d\theta\right])T + (\frac{1}{k_1}\left[l(s)-1\right])M_1. \end{aligned}$$

Theorem 3.2. Let $\alpha = \alpha(s)$ be an arbitrary unit speed curve due to the Bishop frame in Euclidean plane E^2. Position vector and curvature of it satisfy the differential equations of third order

$$\frac{d}{ds}\left[\frac{1}{k_1}\frac{d^2\alpha}{ds^2}\right] - k_1\frac{d\alpha}{ds} = 0.$$

Proof. Let $\alpha = \alpha(s)$ be an arbitrary unit speed curve in Euclidean plane E^2. Then the Bishop derivative formula holds $(3)_1$ in $(3)_2$, we easily have

$$\frac{d}{ds}\left[\frac{1}{k_1}\frac{dT}{ds}\right] + k_1 T = 0. \tag{11}$$

Let $\frac{d\alpha}{ds} = T = \dot{\alpha}$. So, expression of (11) can be written as follows:

$$\frac{d}{ds}\left[\frac{1}{k_1}\frac{d^2\alpha}{ds^2}\right] + k_1\frac{d\alpha}{ds} = 0, \tag{12}$$

formula (12) completes the proof.

Let us solve equation (11) with respect to t. Here we know ,

$$t = (t_1, t_2) = (\dot{\alpha}_1, \dot{\alpha}_2),$$

using the change of variable $\theta = \int_0^s k_1 ds$ in (12) we get

$$\frac{d^2t}{d\theta^2}+\theta=0, \tag{13}$$

or in parametric form it is

$$\frac{d^2t_1}{d\theta^2}+\theta=0, \quad \frac{d^2t_2}{d\theta^2}+\theta=0, \tag{14}$$

as the solution of (14), we obtain

$$\begin{cases} t_1=\eta_1\cosh\theta+\eta_2\sinh\theta, \\ t_2=\eta_3\cosh\theta+\eta_4\sinh\theta, \end{cases} \tag{15}$$

where $\eta_i \in R$ for $1 \le i \le 4$.

4. SPECIAL CURVES IN E^2

In this section we will study some special curves such as Smarandache curves, Circular indicatrices, and curves of constant breadth in Euclidean plane E^2.

4.1. Smarandache Curves

A regular curve in Euclidean plane E^2 whose position vector is composed by Bishop frame vectors on another regular curve, is called a Smarandache curve due to the Bishop frame. In this section we will study TM_1-Smarandache curve as the only Smarandache curve of Euclidean plane E^2**.**

Definition 4.1 (TM_1-**Smarandache curves)**. Let $\alpha=\alpha(s)$ be a unit speed curve due to the Bishop frame in E^2 and $\{T^\alpha, M_1^\alpha\}$ be its moving the Bishop frame, here T^α, M_1^α are the tangent and principal normal vectors of the smarandache curve of the curve α. The curve $\alpha=\alpha(s)$ is said to be TM_1-Smarandache curve whose form is

$$\beta(s^*)=\frac{1}{\sqrt{2}}\left(T^\alpha+M_1^\alpha\right). \tag{16}$$

We can investigate the Bishop invariants of TM_1-Smarandache curves according to $\alpha=\alpha(s)$. Differentiating (16) with respect to s gives us

$$\dot{\beta} = \frac{d\beta}{ds^*}\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\left(k_1^\alpha M_1^\alpha - k_1^\alpha T^\alpha\right). \tag{17}$$

Rearranging of this expression we get

$$T_\beta \frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\left(k_1^\alpha M_1^\alpha - k_1^\alpha T^\alpha\right), \tag{18}$$

by (18) we have

$$\frac{ds^*}{ds} = \left|k_1^\alpha\right|, \tag{19}$$

hence using (18) and (19) we find the tangent vector of the curve β as follows:

$$T_\beta = \mp\frac{\left(M_1^\alpha - T^\alpha\right)}{\sqrt{2}}, \tag{20}$$

and differentiating (20) with respect to s, we have

$$\frac{dT_\beta}{ds^*}\frac{ds^*}{ds} = \mp\frac{\left(-k_1^\alpha T^\alpha - k_1^\alpha M_1^\alpha\right)}{\sqrt{2}}. \tag{21}$$

Substituting (19) in (21), we obtain

$$T_\beta^{'} = \frac{-\left(T^\alpha + M_1^\alpha\right)}{\sqrt{2}}.$$

The curvature and principal normal vector field of the curve β are, respectively,

$$\left\|T_\beta^{'}\right\| = k_{1\beta} = \sqrt{\frac{(T^\alpha)^2 + (M_1^\alpha)^2}{2}},$$

and

$$M_{1\beta} = \frac{-\left(T^\alpha + M_1^\alpha\right)}{\sqrt{(T^\alpha)^2 + (M_1^\alpha)^2}}.$$

4.2. Curves of Constant Breadth

Let $\varphi = \varphi(s)$ and $\varphi^* = \varphi^*(s)$ be simple closed curve due to the Bishop frame in Euclidean plane E^2 These curves will be denoted by C and C^*. The normal plane at every point p on the curve meets the curve at a single point q other then p . We call the point q as the opposite point of p. We consider curves in the class Γ as in Fujivara (1914) having parallel tangents T and T^* in opposite directions at the opposite points φ and φ^* of the curve.

A simple closed curve of constant breadth having parallel tangents in opposite directions at opposite points can be represented with respect to the Bishop frame by the following

$$\varphi^* = \varphi + \lambda T + \mu M_1, \tag{22}$$

where λ and μ are arbitrary functions of s and φ and φ^* which are opposite points.

The vector

$$d = \varphi^* - \varphi$$

is called "the distance vector" between the opposite points of C and C^*. Differentiating (29), and considering Frenet derivative equations (3), we have

$$\frac{d\varphi^*}{ds} = T^* \frac{ds^*}{ds} = T + \frac{d\lambda}{ds}T + \lambda k_1 M_1 + \frac{d\mu}{ds}M_1 - \mu k_1 T$$

Since

$$\frac{d\phi}{ds} = T \text{ and } \frac{d\phi^*}{ds^*} = T^*,$$

and using Bishop derivative formulas, we get

$$T^* \frac{ds^*}{ds} = (1 + \frac{d\lambda}{ds} - \mu k_1)T + (\lambda k_1 + \frac{d\mu}{ds})M_1. \tag{23}$$

Since

$$T^* = -T, \text{and } M_1^* = -M_1, \tag{24}$$

and using (31) in (30), we obtain

$$\frac{ds^*}{ds} = \mu k_1 - \frac{d\lambda}{ds} - 1, \text{ and } \lambda k_1 + \frac{d\mu}{ds} = 0. \tag{25}$$

Let θ be the angle between the tangent vector T at a point $\alpha(s)$ of an oval and a fixed direction, then we have

$$\frac{ds}{d\theta} = \rho = \frac{1}{k_1}, \text{ and } \frac{ds^*}{d\theta} = \rho^* = \frac{1}{k_1^*}. \tag{26}$$

Using (33) in (34), the equation (32) becomes as

$$\begin{cases} \mu - \dfrac{d\lambda}{d\theta} = \rho + \rho^* = f(\theta), \\ \dfrac{d\mu}{d\theta} = -\lambda, \end{cases} \tag{27}$$

eliminating λ in (34) we obtain the linear differential equation of the secon order as

$$\frac{d^2\mu}{d\theta^2} + \mu = f(\theta), \tag{28}$$

where $f(\theta) = \rho + \rho^*$.

By general solution of the equation (35) we find

$$\mu = \sin\theta\left(\int_0^\theta f(t)\cos t dt + l_2\right) - \cos\theta\left(\int_0^\theta f(t)\sin t dt + l_1\right),$$

where l_1, l_2 are scalars. Also using $\lambda = -\dfrac{d\mu}{d\theta}$ in (34) we obtain the value of λ as

$$\lambda = -\cos\theta\left(\int_0^\theta f(t)\cos t dt + l_2\right) - \sin\theta\left(\int_0^\theta f(t)\sin t dt + l_1\right).$$

Hence using (29) the position vector of the curve φ^* is given as follows

$$\begin{aligned} \varphi^* = \varphi + &\left[-\cos\theta\left(\int_0^\theta f(t)\cos t dt + l_2\right) - \sin\theta\left(\int_0^\theta f(t)\sin t dt + l_1\right)\right]T \\ &+ \left[\sin\theta\left(\int_0^\theta f(t)\cos t dt + l_2\right) - \cos\theta\left(\int_0^\theta f(t)\sin t dt + l_1\right)\right]M_1. \end{aligned}$$

If the distance between opposite points of C and C^* is constant, then we can write that

$$\left\|\vec{\varphi}^* - \vec{\varphi}\right\| = -\lambda + \mu = const., \tag{29}$$

and differentiating (36) we have

$$\lambda \frac{d\lambda}{d\theta} + \mu \frac{d\mu}{d\theta} = 0, \tag{30}$$

and also taking the system (34) and (37) together into consideration, we obtain

$$\lambda\left(\frac{d\lambda}{d\theta} - \mu\right) = 0, \tag{31}$$

so we arrive at

$$\lambda = 0 \text{ or } \frac{d\lambda}{d\theta} = \mu \tag{32}$$

Due to the cases in (39), we will consider the conditions below:

If $\lambda = 0$, then from (34) we find that $f(\theta) = const.$**,** and $\mu = const.$

If $\lambda \neq 0 = const.$, and also supposing that $\dfrac{d\lambda}{d\theta} = -\mu$, then we obtain $\mu = 0$.

If $\lambda = c_1, (c_1 \in R)$**,** then the equation (29) tuns into

$$\varphi^* = \varphi + c_1 T. \tag{33}$$

If $\dfrac{d\lambda}{d\theta} = \mu$, then from (34) we have $f(\theta) = 0$, and $\lambda = \int_0^\theta \mu d\theta$.

If $\dfrac{d\lambda}{d\theta} = \mu \neq 0 = c_2 = const.$, then from (34) we obtain $f(\theta) = 0$, and $\lambda = 0$.

Hence the equation (29) becomes as follows:

$$\varphi^* = \varphi + c_2 M_1. \tag{34}$$

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