Karaelmas Fen ve Mühendislik Dergisi
Journal home page: http://fbd.beun.edu.tr

Research Article

On the Periodicies of the Difference Equation $x_{n+1} = x_n x_{n-1}+\alpha$

$x_{n+1} = x_n x_{n-1}+\alpha$ *Fark Denkleminin Periyodikliği Üzerine*

Erkan Taşdemir[1*], Yüksel Soykan[2]

[1]Kirklareli University, Department of Mathematics, Pınarhisar Vocational School of Higher Education, Kirklareli, Turkey
[2]Bülent Ecevit University, Department of Mathematics, Art and Science Faculty, 67100, Zonguldak, Turkey

Abstract

In this paper, we investigate the periodicities and long-term behaviour of the nonlinear difference equation: $x_{n+1}=x_n x_{n-1}+\alpha$, $n\in\mathbb{N}_0$, where the initial conditions x_{-1} and x_0 are real numbers.

2010 AMS-Mathematical Subject Classification Number: 39A22; 39A23

Keywords: Boundedness, Difference equations, Eventual periodicity, Periodicity

Öz

Bu makalede, x_{-1} ve x_0 başlangıç koşulları reel sayılar olmak üzere, $n\in\mathbb{N}_0$ için $x_{n+1}=x_n x_{n-1}+\alpha$, lineer olmayan fark denkleminin periyodikliği ve terimlerinin davranışları incelenmiştir.

2010 AMS-Konu Sınıflandırılması: 39A22; 39A23

Anahtar Kelimeler: Sınırlılık, Fark denklemleri, Eninde sonunda periodiklik, Periodiklik

1. Introduction

In the recent times, nonlinear difference equations have a critical role in the fields of physics, economy, ecology and computational science and engineering, etc. Many researchers have investigated the behavior of the solution of higher order nonlinear difference equations for example:

In [10] Kent et al studied the periodicity of solutions, boundedness of solutions, and existence of unbounded solutions of the nonlinear difference equation

$x_{n+1} = x_n x_{n-1} - 1.$

In [12] Kent et al studied the long-term behavior of solutions of the difference equation

$x_{n+1} = x_{n-1} x_{n-2} - 1.$

In [11] Kent et al investigated the periodicity of solutions, existence of unbounded solutions and converging to the negative equilibrium of the difference equation

$x_{n+1} = x_n x_{n-2} - 1.$

In [13] Kent and Kosmala studied the periodicity and asymptotic periodicity of solutions, as well as the existence of unbounded solutions of the difference equation

$x_{n+1} = x_n x_{n-3} - 1.$

Before the paper [18] of Stevic and Iricanin, the following difference equation $x_{n+1} = x_{n-l} x_{n-k} - 1,\ n\in\mathbb{N}_0$, where $k,l\in\mathbb{N},\ k<l, \gcd(k,l)=1$, and the initial values $x_{-l},...,x_{-2},x_{-1}$ are real numbers, had not been investigated for general case of k and l. In [18], they studied the behavior of solutions of the difference equation of general character, by describing the long-term behavior of the solutions of the difference equation for all values of parameters k and l, where the initial values satisfy the following condition $\min\{x_{-l},...,x_{-2},x_{-1}\}$.

Some difference equations, especially the periodicity, boundedness and some other properties of higher order nonlinear difference equations have been investigated by many authors, see [1]-[20].

Ladas [2] investigated the stability of the equilibrium points

*Corresponding Author: erkantasdemir@hotmail.com

Received / Geliş tarihi : 01.04.2016
Accepted / Kabul tarihi : 20.07.2016

and the long-term behavior of solution of second-order difference equation

$$x_{n+1} = x_n x_{n-1} + \alpha, n = 0, 1, 2, ..., \tag{1}$$

where the initial conditions x_{-1} and x_0 are real numbers and $\alpha \in \mathbb{R}$. In this paper, we examine the periodic behaviors of solutions and dependence of such behaviors on initial conditions of the Eq.(1).

2. The Equilibria of Eq.(1)

In this section, we investigate that Eq.(1) have exactly two equilibria and nontrivial-periodic solutions.

After solving the equation $\bar{x}^2 - \bar{x} + \alpha = 0$, we find that Eq.(1) has exactly two equilibria, which $\bar{x}_1$ is negative number and $\bar{x}_2$ is positive number together:

$$\bar{x}_{1,2} = \frac{1 \pm \sqrt{1 - 4\alpha}}{2}. \tag{2}$$

Note that there are three cases for (2):

Case 1. $\bar{x}$ is complex number if $\alpha > \frac{1}{4}$,

Case 2. $\bar{x}_1$ and $\bar{x}_2$ are real numbers if $\alpha < \frac{1}{4}$

Case 3. $\bar{x} = \frac{1}{2}$ is the unique equilibrium if $\alpha = \frac{1}{4}$.

3. The Periodic Solutions of Eq.(1)

Now, we study the existence of periodic or eventually periodic solutions of Eq.(1).

Theorem 1 There are no eventually constant solutions of Eq.(1).

Proof. If $\{x_n\}_{n=-1}^{\infty}$ is eventually constant solutions of Eq.(1), then $x_N = x_{N+1} = \bar{x}$ for some $N \geq 0$, where $\bar{x}$ is an equilibrium point. Therefore from $x_{N+1} = x_N x_{N-1} + \alpha$, it follows that

$$x_{N-1} = \frac{x_{N+1} - \alpha}{x_N} = \frac{\bar{x} - \alpha}{\bar{x}} = \bar{x}.$$

Repeating this procedure, we obtain $x_n = \bar{x}$, for $-1 \leq n \leq N+1$ as claimed.

Theorem 2 Difference equation (1) has no nontrivial period two solutions nor eventually period two solutions.

Proof. Suppose that $x_N = x_{N+2k}$ and $x_{N+1} = x_{N+2k+1}$, for every $k \in \mathbb{N}_0$, and some $N \geq -1$, with $x_N \neq x_{N+1}$. Therefore, we have

$$\begin{aligned} x_{N+4} &= x_{N+3} x_{N+2} + \alpha \\ &= x_{N+1} x_{N+2} + \alpha = x_{N+3} \\ &= x_{N+1} x_N + \alpha = x_{N+2} \\ &= x_{N-1} x_N + \alpha = x_{N+1}. \end{aligned}$$

From this and since $x_{N+4} = x_N$, we obtain a contradiction which finishes the proof of the result.

The following result shows that there exists exactly three periodic solution of Eq.(1) with minimal period three and gives a description of each.

Theorem 3 There exists exactly three periodic solution of Eq.(1) with minimal period three. They are given by the three pairs of initial conditions

$x_{-1} = -1, x_0 = -1;$

$x_{-1} = -1, x_0 = \alpha + 1;$

and

$x_{-1} = \alpha + 1, x_0 = -1;$

where $\alpha \neq 0$.

Proof. The case $\alpha = -1$ was investigated in [1]. Now, we can write terms of a period-three solution of Eq.(1) as

$x_{-1} = a,$

$x_0 = b,$

$x_1 = ab + \alpha$

and

$x_2 = (ab + \alpha) b + \alpha = a$

$x_3 = a(ab + \alpha) + \alpha = b$

so

$ab^2 + \alpha b + \alpha - a = 0,$

$ba^2 + \alpha a + \alpha - b = 0.$

Thus, this is indeed a solution of period three if the system below is satisfied.

$$(b+1)(ab - a + \alpha) = 0 \tag{3}$$

$$(a+1)(ab - b + \alpha) = 0. \tag{4}$$

Therefore, we obtain three cases after solving (3) and (4),

Case 1. Suppose that $b + 1 = 0$. From (4) then $(a+1)(ab - b + \alpha) = 0$ implies that $ab - b + \alpha = 0$ or $a + 1 = 0$ and hence $a = -1$ or $a = \alpha + 1$.

Case 2. Suppose that $a + 1 = 0$ From (3) then $(b+1)(ab - a + \alpha) = 0$ implies that $ab - a + \alpha = 0$ or $b + 1 = 0$ and hence $b = -1$ or $b = \alpha + 1$.

Case 3. Suppose that $a + 1 \neq 0$ and $b + 1 \neq 0$ such that

$$ab - a + \alpha = 0 \tag{5}$$

$$ab - b + \alpha = 0. \tag{6}$$

Therefore, after solving (5)-(6) we find that

$a = \bar{x}_1$ and $b = \bar{x}_1$

or

$a = \bar{x}_2$ and $b = \bar{x}_2$.

In this case, there are no periodic solutions of Eq.(1) with prime period three, because it has trivial solutions.

Hence, there exists exactly three periodic solutions with minimal period three of Eq.(1) given by

$$\begin{aligned} &x_{-1} = -1, x_0 = -1, x_1 = \alpha + 1, \ldots \\ &x_{-1} = \alpha + 1, x_0 = -1, x_1 = -1, \ldots \\ &x_{-1} = -1, x_0 = \alpha + 1, x_1 = -1, \ldots \end{aligned} \tag{7}$$

as claimed. In addition, the graphs of Eq.(1) are presented below where the initial conditions are given in (7).

In the sequel, we will refer to any one of these three periodic solutions of Eq.(1) as

$$\ldots, \alpha + 1, -1, -1, \alpha + 1, -1, -1, \ldots \tag{8}$$

Theorem 4 Difference equation (1) has no nontrivial period-four solutions.

Proof. Let $\{x_n\}_{n=-1}^{\infty}$ be a period-four solution of Eq.(1). Then $x_{4n-1} = a, x_{4n} = b, x_{4n+1} = c$ and $x_{4n+2} = d$ for every $n \in \mathbb{N}_0$ and some $a,b,c,d \in \mathbb{R}$ such that at least two of them are different. Therefore by direct calculation we have

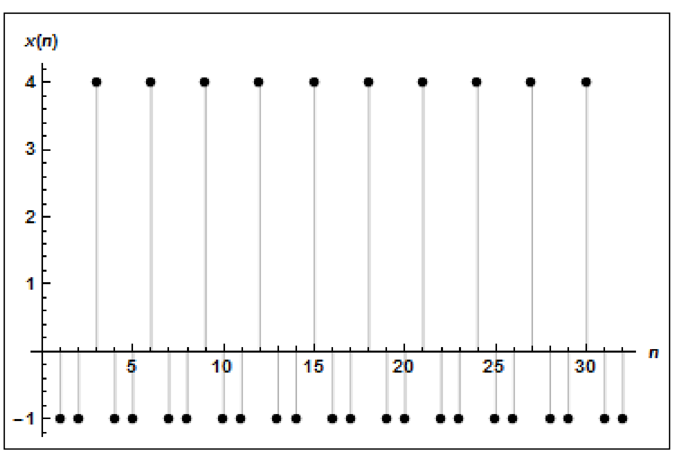
$$x_1 = x_0 x_{-1} + \alpha = ab + \alpha = c \tag{9}$$

$$x_2 = x_1 x_0 + \alpha = b(ab + \alpha) + \alpha = d \tag{10}$$

$$x_3 = x_2 x_1 + \alpha = cd + \alpha = a \tag{11}$$

$$x_4 = x_3 x_2 + \alpha = ad + \alpha + \alpha = b. \tag{12}$$

Thus, from (9)-(12), we have two cases,



Graph 1: $x_{n+1} = x_n x_{n-1} + 3; x_{-1} = -1, x_0 = -1$ and α = 3.

Case 1. If $\alpha \neq 0$, then $a = b = c = d = \bar{x}_1$ and $a = b = c = d = \bar{x}_2$,

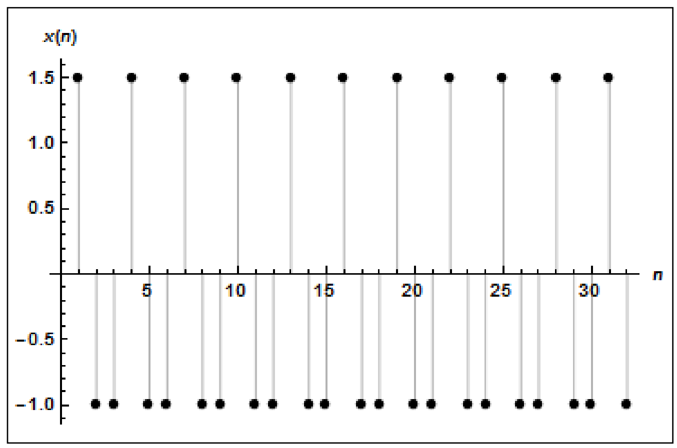
Case 2. If $\alpha = 0$, then $a = b = c = d = 0$ and $a = b = c = d = 1$.

Consequently, there are no periodic solutions of Eq.(1) with period-four, because it's trivial solutions or non-periodic solutions.

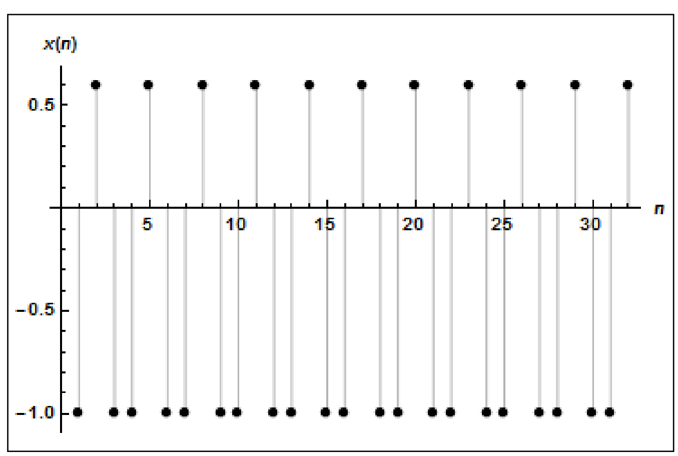
Theorem 5 Difference equation (1) has no nontrivial period-five solutions.

Proof. Let $\{x_n\}_{n=-1}^{\infty}$ be a period-five solution of Eq.(1). Then $x_{5n-1} = a, x_{5n} = b, x_{5n+1} = c, x_{5n+2} = d$ and $x_{5n+3} = e$ for every $n \in \mathbb{N}_0$ and some $a,b,c,d,e \in \mathbb{R}$ such that at least two of them are different. Therefore by direct calculation we have

$$x_1 = x_0 x_{-1} + \alpha = ab + \alpha = c \tag{13}$$



Graph 2: $x_{n+1} = x_n x_{n-1} + \frac{1}{2}; x_{-1} = \alpha + 1 = \frac{3}{2}, x_0 = -1$ and $\alpha = \frac{1}{2}$.



Graph 3: $x_{n+1} = x_n x_{n-1} - \frac{2}{5}; x_{-1} = -1, x_0 = \alpha + 1 = \frac{3}{5}$ and $\alpha = -\frac{2}{5}$.

$$x_2 = x_1 x_0 + \alpha = bc + \alpha = d \quad (14)$$

$$x_3 = x_2 x_1 + \alpha = cd + \alpha = e \quad (15)$$

$$x_4 = x_3 x_2 + \alpha = de + \alpha = a \quad (16)$$

$$x_5 = x_4 x_3 + \alpha = ae + \alpha = b \quad (17)$$

Then, from (13)-(17), $a = b = c = d = \bar{x}_1$ and $a = b = c = d = \bar{x}_2$. So, there are no periodic solutions of Eq.(1) with period-five, because it's trivial solution.

4. The Long-Term Behaviour of Eq.(1)

In this section, we find sets of initial conditions of Eq.(1) for which unbounded solutions exist. The following theorem take care of long-term behaviour of Eq.(1).

Theorem 6 Let $\alpha > \frac{1}{4}$ and $x_{-1}, x_0 < -1$. Then

$$0 < x_1 < x_4 < x_7 < ...$$
$$... < x_8 < x_6 < x_5 < x_3 < x_2 < x_0 < -1$$

and subsequences $\{x_{3n}\}_{n=0}^{\infty}, \{x_{3n-1}\}_{n=0}^{\infty}$ tend to $-\infty$, $\{x_{3n+1}\}_{n=0}^{\infty}$ tends to $+\infty$.

Proof. We first have, $x_1 = x_0 x_{x-1} + \alpha > 0$. We will prove that, $x_2 < x_0 < -1$.

Since $(x_0+1)^2 > 0$, we have that $x_0^2 + 2x_0 + 1 > 0$. Thus, $1 + \frac{2}{x_0} + \frac{1}{x_0^2} > 0$ so $\frac{2}{x_0} + \frac{1}{x_0^2} > -1$. Since $x_{-1} < -1$, we must have, $x_{-1} < \frac{2}{x_0} + \frac{1}{x_0^2}$. Thus $x_{-1} x_0 > 2 + \frac{1}{x_0}$ and $x_{-1} x_0 + \alpha > 2 + \alpha + \frac{1}{x_0}$. Therefore $x_1 > 2 + \alpha + \frac{1}{x_0}$. It follows that

$$x_1 x_0 < 2x_0 + \alpha x_0 + 1$$
$$x_1 x_0 + \alpha < 2x_0 + \alpha x_0 + 1 + \alpha < x_0$$
$$x_2 < x_0 < -1.$$

Next, we show that x_3 is not only less than -1, but it is also less then x_2. To show this, we note that since $x_2 < x_0$ and $x_1 = x_0 x_{-1} + \alpha > 0$ we have $x_2 x_1 < x_0 x_1$. This gives $x_2 x_1 + \alpha < x_0 x_1 + \alpha$, and hence $x_3 < x_2 < x_0 < -1$.

We want to show $x_4 > x_1$. To show this, we start by observing that $x_3 < -1$ and so $x_3 < x_1$. Therefore, since $x_3 = x_2 x_1 + \alpha < x_1$, we obtain with $x_2 + 1 < 0$,

$$x_2 x_1 - x_1 + \alpha < 0$$
$$x_1(x_2 - 1) + \alpha < 0$$
$$x_1(x_2 - 1) < -\alpha$$
$$x_1(x_2 - 1)(x_2 + 1) > -\alpha(x_2 + 1)$$
$$x_1 x_2^2 - x_1 > -\alpha x_2 - \alpha$$
$$x_1 x_2^2 + \alpha x_2 > x_1 - \alpha$$
$$x_2(x_2 x_1 + \alpha) > x_1 - \alpha$$
$$x_2 x_3 + \alpha > x_1$$
$$x_4 > x_1.$$

By induction, it can be proved that,

$$0 < x_1 < x_4 < x_7 < ...$$
$$... < x_8 < x_6 < x_5 < x_3 < x_2 < x_0 < -1.$$

In other words, $\{x_{3n+1}\}_{n=0}^{\infty}$ is a positive increasing subsequence and $\{x_{3n}\}_{n=0}^{\infty}$ and $\{x_{3n+2}\}_{n=0}^{\infty}$ are negative decreasing subsequences.

Next we verify that these subsequences are unbounded and thus that our solution is unbounded. Suppose that the two decreasing sequences, $\{x_{3n}\}_{n=0}^{\infty}$ and $\{x_{3n+2}\}_{n=0}^{\infty}$ are bounded from below. Then they each must converge to a finite limit (which is the same finite limit and less then $\bar{x}_2$). But by Eq.(1), the third increasing subsequence $\{x_{3n+1}\}_{n=0}^{\infty}$ must also converge to a finite limit (which is positive), where,

$$x_{3n+1} = x_{3n} x_{3n-1} + \alpha = |x_{3n}| \cdot |x_{3n-1}| + \alpha.$$

This is impossible because there are no periodic solution with minimal period two (see Section 3, theorem 2). So $\{x_{3n}\}_{n=0}^{\infty}$ and $\{x_{3n+2}\}_{n=0}^{\infty}$ are unbounded (They tend to $-\infty$) and thus $\{x_{3n+1}\}_{n=0}^{\infty}$ is also unbounded (it tends to $+\infty$) by Eq.(1) again.

5. References

Amleh, AM., Camouzis, E., Ladas, G. 2008. On the dynamics of a rational difference equation, Part 2. *IJDE*, 3(2), 195-225.

Amleh, AM., Camouzis, E., Ladas, G. 2008. On the dynamics of a rational difference equation, Part I. *IJDE*, 3(1), 1-35.

Elaydi, S. 1996. *An introduction to difference equations*. New York: Springer.

Elsayed, EM., El-Dessoky, MM. 2013. Dynamics and global behavior for a fourth-order rational difference equation. *Hacet. J. Math. Stat.*, *42*(5), 479-494.

Gümüş, M. 2013. The Periodicity of Positive Solutions of the Nonlinear Difference Equation . *Discrete Dyn. Nat. Soc.*, *2013*, 1-3. doi:10.1155/2013/742912

Gümüş, M., Öcalan, Ö. 2012. Some Notes on the Difference Equation *Discrete Dyn. Nat. Soc.*, *2012*, 1-12. doi:10.1155/2012/258502

Gümüş, M., Öcalan, Ö. 2014. Global Asymptotic Stability of a Nonautonomous Difference Equation. *J. Appl. Math.*, *2014*, 1-5. doi:10.1155/2014/395954

Gümüş, M., Öcalan, Ö., Felah, NB. 2012. On the Dynamics of the Recursive Sequence. *Discrete Dyn. Nat. Soc.*, *2012*, 1-11. doi:10.1155/2012/241303

Iričanin, B., Stević, S. 2009. Eventually constant solutions of a rational difference equation. *Appl. Math. Comput.*, *215*(2), 854-856. doi:10.1016/j.amc.2009.05.044

Kent, CM., Kosmala, W., Radin, MA., Stević, S. 2010. Solutions of the Difference Equation . *Abstr. Appl. Anal.*, *2010*, 1-13. doi:10.1155/2010/469683

Kent, CM., Kosmala, W., Stević, S. 2011. On the Difference Equation. *Abstr. Appl. Anal.*, *2011*, 1-15. doi:10.1155/2011/815285

Kent, CM., Kosmala, W., Stević, S. 2010. Long-Term Behavior of Solutions of the Difference Equation . *Abstr. Appl. Anal.*, *2010*, 1-17. doi:10.1155/2010/152378

Kent, CM., Kosmala, W. 2011. On the Nature of Solutions of the Difference Equation . *IJNAA*, *2*(2), 24-43.

Kosmala, W. 2011. A period 5 difference equation. *IJNAA*, *2*(1), 82-84.

Öcalan, Ö., Ogünmez, H., Gümüş, M. 2014. Global behavior test for a nonlinear difference equation with a period-two coefficient. *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.*, *21*(3-4), 307-316.

Stevic, S. 2008. On the difference equation . *Comput. Math. Appl.*, *56*, 1159-1171.

Stević, S. 2008. Nontrivial solutions of a higher-order rational difference equation. *Math. Notes*, *84*(5-6), 718-724. doi:10.1134/s0001434608110138

Stević, S., Iričanin, B. 2011. Unbounded Solutions of the Difference Equation. *Abstr. Appl. Anal.*, *2011*, 1-8. doi:10.1155/2011/561682

Stević, S., Alghamdi, MA., Alotaibi, A. 2015. Boundedness character of the recursive sequence . *Appl. Math. Lett.*, *50*, 83-90. doi:10.1016/j.aml.2015.06.006

Stevic, S., Diblik, J., Iricanin, B., Smarda, Z. 2014. Solvability of nonlinear difference equations of fourth order. *EJDE*, *2014*(264), 1-14.