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On ternary biquadratic Diophantine equation $11(x^2 - y^2) + 3(x + y) = 10z^4$

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Abstract: We obtain infinitely many non-zero integer triples (x, y, z) satisfying the non-homogeneous bi-quadratic equation with three unknowns $11(x^2 - y^2) + 3(x + y) = 10z^4$. Various interesting properties among the values of x, y, z are presented. Some relations between the solutions and special numbers are exhibited.
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1 Introduction

The theory of Diophantine equations offers a rich variety of fascinating problems. Since antiquity, mathematicians exhibit great interest in homogeneous and non-homogeneous bi-quadratic Diophantine equations. In this context, one may refer our references for a variety of problems on the bi-quadratic Diophantine equations with three variables and also for bi-quadratic equations with four unknowns studied on their integral solutions. This communication concerns a yet another interesting ternary bi-quadratic equation given by

$11(x^2-y^2)+3(x+y)=10z^4$ and is analyzed for its non-zero distinct integer solutions. Also, a few interesting relations between the solutions and special numbers are presented.

Definition: A *nasty number* is a positive integer with at least four different factors such that the difference between the numbers in one pair of factors is equal to the sum of the numbers of another pair and the product of each pair is equal to the number. Thus a positive integer n is a nasty number, if $m = u * v = t * s$ and $u + v = t - s$ where $u, v, t, s \in \mathbb{Z}^+$.

Lemma: Properties of nasty numbers are given by following expressions:

1. If t is a nasty number, then clearly $t*s^2$ is also a nasty number for every non zero integral value of t.
2. If four positive integers $\alpha, \beta, \gamma, \delta$ such that α, β, γ are in arithmetic progression with δ as their common difference, then $m = \alpha * \beta * \gamma * \delta$ is a nasty number.
3. Every integer u of the form $6.(1^2 + 2^2 + 3^2 + \cdots + k^2)$ is a nasty number.
4. Every integer u of the form $6.[1^2 + 3^2 + \cdots + (2k-1)^2]$ is a nasty number.

Theorem (Brahmagupta's Lemma): If (a_1, b_1) is a solution of $Da^2 + m_1 = b^2$ and (a_2, b_2) is a solution of $Da^2 + m_2 = b^2$, then $(a_1b_2 + a_2b_1, b_2b_1 + Da_1a_2)$ and $(a_1b_2 - a_2b_1, b_2b_1 - Da_1a_2)$ are solutions of $Da^2 + m_1m_2 = b^2$.

Notations:

- Polygonal number of rank n with size m is defined by
$$t_{m,n} = n\left[1+\frac{(n-1)(m-2)}{2}\right];$$
- Pyramidal number of rank n with size m is defined by
$$P_n^m = \frac{1}{6}\left[n(n+1)\right]\left[(m-2)n+(5-m)\right];$$
- Centered Pyramidal number of rank n with size m is given by
$$CP_{m,n} = \frac{m(n-1)n(n+1)+6n}{6};$$
- Stella Octangula number of rank n is given by $SO_n = 2n^3 - n$;
- Gnomonic number of rank n is formulated by $GNO_n = 2n-1$;
- Rhombic Dodagonal number of rank n is defined by $R_n = 4n^3 - 6n^2 + 4n - 1$;
- Star number of rank n is formulated by $S_n = 6n^2 - 6n + 1$.

2 Method of analysis

The non-homogeneous bi-quadratic Diophantine equation to be solved for its distinct non-zero integral solution is

$$11(x^2-y^2)+3(x+y)=10z^4. \tag{1}$$

Different patterns of solutions for (1) are illustrated below.

2.1. Pattern 1

Introduction of the transformations

$$x = 5u^2 + v^2, \quad y = 5u^2 - v^2, \quad z = u \tag{2}$$

in (1) leads to

$$u^2 = 22v^2 + 3. \tag{3}$$

The least positive solution of (3) is $v_0 = 1, u_0 = 5$. To find the other solutions of (3), consider the positive Pell equation

$$u^2 = 22v^2 + 1,$$

whose general solution $(\tilde{u}_n, \tilde{v}_n)$ is given by

$$\tilde{u}_n = \frac{1}{2}\left[\left(197 + 42\sqrt{22}\right)^{n+1} + \left(197 - 42\sqrt{22}\right)^{n+1}\right] = \frac{1}{2} f_n,$$

$$\tilde{v}_n = \frac{1}{2\sqrt{22}}\left[\left(197 + 42\sqrt{22}\right)^{n+1} - \left(197 - 42\sqrt{22}\right)^{n+1}\right] = \frac{1}{2\sqrt{22}} g_n,$$

$n = -1, 0, 1, 2, \ldots$. Applying the Brahmagupta lemma between the solutions (u_0, v_0) and $(\tilde{u}_n, \tilde{v}_n)$, the sequence of integer solutions to (3) are given by

$$u_{n+1} = \frac{5}{2} f_n + \frac{\sqrt{22}}{2} g_n,$$

$$v_{n+1} = \frac{1}{2} f_n + \frac{5\sqrt{22}}{44} g_n.$$

Employing (2), the corresponding non-zero integer solutions to (1) are given by

$$x_{n+1} = \frac{63}{2} f_n^2 + \frac{2445}{88} g_n^2 + \frac{555\sqrt{22}}{44} f_n g_n,$$

$$y_{n+1} = 31 f_n^2 + \frac{2395}{88} g_n^2 + \frac{545\sqrt{22}}{44} f_n g_n,$$

$$z_{n+1} = \frac{5}{2} f_n + \frac{\sqrt{22}}{2} g_n.$$

A few numerical examples are illustrated in the Table 1 below.

n	x_{n+1}	y_{n+1}	z_{n+1}
-1	126	124	5
0	18387054	18055756	1909
1	2854294786854	2802866051956	752141

Table 1. Numerical examples

Properties of Pattern 1:

- $x_{n+1} + y_{n+1} = 10z_{n+1}^2$;
- $x_{n+1} + y_{n+1} - 9z_{n+1} = t_{22,z_{n+1}}$;
- $(x_{n+1} + y_{n+1})(z_{n+1} + 1) = 20P_{z_{n+1}}^5$;
- $(x_{n+1} + y_{n+1} - 10)\, z_{n+1} = 60P_{z_{n+1}-1}^3$;
- $(x_{n+1} + y_{n+1} - 10)\, z_{n+1} = 30(P_{z_{n+1}}^4 - t_{3,z_{n+1}})$;
- $22y_{n+1} - 109z_{n+1}^2 = 3$;
- $111y_{n+1} - 109x_{n+1} = 30$;
- $111z_{n+1}^2 - 22x_{n+1} = 3$;
- $3(x_{n+1} - y_{n+1})$ is a nasty number;
- Each of the following expression represents a bi-quadratic integer:

$$25z_{n+1}^4 - x_{n+1}y_{n+1},$$
$$\frac{x_{n+1}^2 + y_{n+1}^2 - 50z_{n+1}^4}{2}.$$

2.2. Pattern 2

Introducing the linear transformations

$$x = 5u + v, \quad y = 5u - v, \quad z = u \tag{4}$$

in (1), it becomes

$$u^3 = 22v + 3 \tag{5}$$

The choice

$$u = 22k + 9 \tag{6}$$

in (5) leads to

$$v = 484k^3 + 594k^2 + 243k + 33.$$

In view of (4), the corresponding non-zero distinct integral solutions to (1) are given by

$$x(k) = 484k^3 + 594k^2 + 353k + 78,$$
$$y(k) = -484k^3 - 594k^2 - 133k + 12,$$
$$z(k) = 22k + 9.$$

Properties of Pattern 2:

- $y(k) + z(k) + 484CP_{6,k} + 594t_{4,k} + 54GNO_k \equiv 0 \pmod 3$;
- $x(k) + z(k) - 242SO_k - 99t_{14,k} - 555GNO_k \equiv 0 \pmod 2$;
- $(x(k) + y(k))\, z^2(k) - 5SO_{z(k)} \equiv 0 \pmod 5$.

2.3. Pattern 3

Instead of (4), if we consider the linear transformations

$$x = u + v\,, \quad y = u - v\,, \quad z = u \tag{7}$$

in (1), it becomes

$$5u^3 = 22v + 3 \tag{8}$$

The choice

$$u = 22k + 3 \tag{9}$$

in (8) leads to

$$v = 2420k^3 + 990k^2 + 135k + 6$$

Substituting the values of *u, v* in (7), the corresponding non-zero distinct integral solutions to (1) are given by

$$x(k) = 2420k^3 + 990k^2 + 157k + 9$$

$$y(k) = -2420k^3 - 990k^2 - 113k - 3$$

$$z(k) = 22k + 3$$

Properties of Pattern 3:

- $y(k) + 2420CP_{6,k} + 110t_{20,k} \equiv 0 (\bmod 3)$;
- $x(k) - y(k) - 2420SO_k - 1980t_{4,k} \equiv 0 (\bmod 2)$;
- $3\,(x(k) + y(k))\,z(k)$ is a nasty number.

2.4. Pattern 4

By introducing the linear transformation

$$x = y + t \tag{10}$$

in (1), it results in

$$11t^2 + (22y + 3)t + 6y - 10z^4 = 0 \tag{11}$$

which is a quadratic in *t* and solving for *t*

$$t = \frac{1}{22}\left(-22y - 3 \pm \sqrt{(22y - 3)^2 + 440z^4}\right) \tag{12}$$

Let $\alpha^2 = 440z^4 + (22y - 3)^2$ and it is satisfied by

$$\left.\begin{aligned} z^2 &= 2rs \\ 22y - 3 &= 440r^2 - s^2 \\ \alpha &= 440r^2 + s^2 \end{aligned}\right\} \tag{13}$$

As our interest is on finding the integer solutions, replace *r* by 2*s* and *s* by 22*k* + 5 in (12) and (13), it is seen that

$$\left.\begin{aligned} y &= 38698k^2 + 17590k + 1999 \\ z &= 44k + 10 \end{aligned}\right\} \tag{13}$$

$$t = 44k^2 + 20k + 2,$$

the negative sign before the square root in (12) is not admissible. In view of (10), we obtain

$$x = 38742k^2 + 17610k + 2001. \tag{15}$$

Thus, (14) and (15) represents non-zero distinct integer solutions to (1).

Properties of Pattern 4:

- $x(k) - z(k) - 6457S_k \equiv 0(\bmod 2)$;
- $y(k) - 3z(k) - 1759t_{46,k} - 27197GNO_k \equiv 0(\bmod 3)$;
- $x(k) - y(k) - 44t_{4,k} \equiv 0(\bmod 2)$.

3 Conclusion

In this paper, we have made an attempt to find different patterns of non-zero distinct integer solutions to the bi-quadratic equation with three unknowns given by $11(x^2 - y^2) + 3(x + y) = 10z^4$. As bi-quadratic equations are rich in variety, one may search for integer solutions to other choices of bi-quadratic and higher order equations with multivariaties along with suitable properties.

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