ORIGINAL PAPER

QUALITATIVE BEHAVIOURS OF A SYSTEM OF NONLINEAR DIFFERENCE EQUATIONS

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Manuscript received: 17.09.2020; Accepted paper: 22.11.2020;
Published online: 30.03.2021.

Abstract. *The paper aims to study the dynamics of a system of nonlinear difference equations* $x_{n+1} = x_{n-1}y_n + A, y_{n+1} = y_{n-1}x_n + A$ *where* A *is real number. We especially investigate the stability of equilibrium points, convergence of equilibrium points, existence of periodic solutions, and existence of bounded solutions of related system. Moreover, we present some numerical examples to verify the theoretical results*

Keywords: *difference equations; dynamical systems; stability; global stability; periodicity; boundedness.*

1. INTRODUCTION

The theory of discrete dynamical systems or difference equations has a huge importance for the applied sciences based on mathematical modelling. Many applied sciences like biology, need to discrete variables for mathematical modeling. The mathematical models produced by the difference equations have been applied to various disciplines by many researchers in the field of applied sciences. For all these reasons, there is an increasing interest in difference equations in recent years. This interest especially focuses on the dynamic behaviors of dynamical systems. Moreover, many books and papers have been published related to difference equations and dynamical systems in literature, we refer to [1-33].

During the few last years, many authors studied the dynamics of solutions of the system of difference equations, for example:

In [21], Papaschinopoulos et al. studied the oscillatory behavior, the periodicity and asymptotic behavior of the positive solutions of system of two nonlinear difference equations

$$x_{n+1} = A + \frac{x_{n-1}}{y_n}, y_{n+1} = A + \frac{y_{n-1}}{x_n}.$$

In [33], Zhang et al. investigated the boundedness, persistence and global asymptotic stability of positive solutions of system of difference equations

$$x_{n+1} = A + \frac{y_{n-m}}{x_n}, y_{n+1} = A + \frac{x_{n-m}}{y_n}.$$

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In [18], Kurbanli et al. considered the following system of difference equations

$$x_{n+1}=\frac{x_{n-1}}{y_nx_{n-1}+1},y_{n+1}=\frac{y_{n-1}}{x_ny_{n-1}+1}.$$

In [5] and [6], Clark et al. studied the global asymptotic stability of following system of difference equations

$$x_{n+1}=\frac{x_n}{a+cy_n},y_{n+1}=\frac{y_n}{b+dx_n}.$$

In [13], Kent et al. studied long-term behaviour of solutions of difference equation

$$x_{n+1}=x_nx_{n-1}-1.$$

They examined the two and three periodic solutions, eventually periodic solutions, invariant interval and unbounded solutions of related equation. In addition, in [31], Wang et al. investigated the convergence of negative equilibrium point of equation. Moreover, in [19], Liu et al. handled related difference equation.

In [2], Amleh et al. investigated the stability of following difference equations

$$x_{n+1}=x_nx_{n-1}+\alpha.$$

Moreover, In [30], Taşdemir et al. studied periodic solutions and unbounded solutions of related difference equations.

Following these studies, in [27] Taşdemir investigated dynamics of the following system of nonlinear difference equations

$$x_{n+1}=x_{n-1}y_n-1,y_{n+1}=y_{n-1}x_n-1.$$

The author especially examined the existence of bounded solutions, stability of two equilibrium points of system, the convergence of negative equilibrium point, existence of periodic solutions of related system.

Motivated by this studies, we study the dynamics of a system of nonlinear difference equations

$$x_{n+1}=x_{n-1}y_n+A,y_{n+1}=y_{n-1}x_n+A,n=0,1,\ldots, \tag{1.1}$$

where A is real number. We especially investigate the stability of equilibrium points, convergence of equilibrium points, existence of periodic solutions, and existence of bounded solutions of related system.

Although these form of the difference equations and the systems given above are seemingly simple, the behaviors of solutions of these equations or systems is quite different from each other.

2. MAIN RESULTS

Firstly, we reveal the equilibrium points of system (1.1). We also deal with the conditions of the elements of equilibrium points of the related system for intervals of A.

Lemma 1. There are two equilibrium points of System (1.1) as follows:

$$(\bar{x}_1, \bar{y}_1) = \left(\frac{1-\sqrt{1-4A}}{2}, \frac{1-\sqrt{1-4A}}{2}\right),$$

$$(\bar{x}_2, \bar{y}_2) = \left(\frac{1+\sqrt{1-4A}}{2}, \frac{1+\sqrt{1-4A}}{2}\right).$$

Note that, in case of $A = -1$, the components of second equilibrium point equal to Golden Ratio that is $\frac{1+\sqrt{5}}{2} \approx 1.618$.

Lemma 2. The following statements are true:

i. If $A < \frac{1}{4}$, then the elements of two equilibrium points of system (1.1) are real numbers.

ii. If $A = \frac{1}{4}$, then system (1.1) has a unique equilibrium point such as $(\bar{x}_1, \bar{y}_1) = (\bar{x}_2, \bar{y}_2) = \left(\frac{1}{2}, \frac{1}{2}\right)$.

iii. If $A > \frac{1}{4}$, then the elements of two equilibrium points of system (1.1) are complex numbers.

iv. If $A < 0$, then the elements of equilibrium point $(\bar{x}_1, \bar{y}_1)$ of system (1.1) are negative numbers.

Now, we investigate the two and three periodic solutions of system (1.1). Additionally, we determine the initial values for these periodic solutions.

Theorem 1. Assume that $A \leq \frac{1}{4}$. If the initial values are $x_{-1} = \bar{x}_1$, $x_0 = \bar{x}_2$, $y_{-1} = \bar{y}_2$, $y_0 = \bar{y}_1$ or $x_{-1} = \bar{x}_2$, $x_0 = \bar{x}_1$, $y_{-1} = \bar{y}_1$, $y_0 = \bar{y}_2$, then system (1.1) has two periodic solutions such as

$$\{\cdots, (\bar{x}_1, \bar{y}_2), (\bar{x}_2, \bar{y}_1), (\bar{x}_1, \bar{y}_2), \cdots\}.$$

Proof: Let $\{(x_n, y_n)\}_{n=-1}^{\infty}$ be a two periodic solution of system (1.1). Let a, b, c, d are real numbers such that $a \neq b$ and $c \neq d$. Suppose that

$$x_{2n} = a, x_{2n-1} = b, y_{2n} = c, y_{2n-1} = d,$$

for all $n \geq 0$. Hence, we obtain from system (1.1):

$$x_{2n+1} = x_{2n-1}y_{2n} + A, \tag{2.1}$$

$$x_{2n+2} = x_{2n}y_{2n+1} + A, \tag{2.2}$$

$$y_{2n+1} = y_{2n-1}x_{2n} + A, \tag{2.3}$$

$$y_{2n+2} = y_{2n}x_{2n+1} + A. \tag{2.4}$$

Therefore, we have from (2.1)-(2.4):

$$b \cdot c + A = b, \tag{2.5}$$

$$d \cdot a + A = d, \tag{2.6}$$

$$a \cdot d + A = a, \tag{2.7}$$

$$c \cdot b + A = c. \tag{2.8}$$

From (2.5)-(2.8), we obtain four cases such that:

$$a = \bar{x}_1, b = \bar{x}_2, c = \bar{y}_2, d = \bar{y}_1, \tag{2.9}$$

$$a = \bar{x}_2, b = \bar{x}_1, c = \bar{y}_1, d = \bar{y}_2, \tag{2.10}$$

$$a = b = \bar{x}_1, c = d = \bar{y}_1, \tag{2.11}$$

$$a = b = \bar{x}_2, c = d = \bar{y}_2. \tag{2.12}$$

Consequently, (2.9) and (2.10) are two periodic solutions of system (1.1) but (2.11) and (2.12) are trivial solutions of system (1.1). The proof completed.

Lemma 3. Let $A \le \frac{1}{4}$. From (2.9) and (2.10), system (1.1) has a periodic cycle with period two as:

$$\{\cdots, (\bar{x}_1, \bar{y}_2), (\bar{x}_2, \bar{y}_1), (\bar{x}_1, \bar{y}_2), \cdots\}$$

Proof: Let $x_{-1} = \bar{x}_1, x_0 = \bar{x}_2, y_{-1} = \bar{y}_2, y_0 = \bar{y}_1$. Thus, from system (1.1) we get that:

$$x_1 = x_{-1}y_0 + A = \bar{x}_1 \cdot \bar{y}_1 + A = \bar{x}_1,$$

$$y_1 = y_{-1}x_0 + A = \bar{y}_2 \cdot \bar{x}_2 + A = \bar{y}_2,$$

$$x_2 = x_0y_1 + A = \bar{x}_2 \cdot \bar{y}_2 + A = \bar{x}_2,$$

$$y_2 = y_0x_1 + A = \bar{y}_1 \cdot \bar{x}_1 + A = \bar{y}_1.$$

So, two periodic cycle of system (1.1) completed.

Theorem 2. System (1.1) has three periodic solutions, if and only if the initial values are

$$x_{-1} = -1, x_0 = -1, y_{-1} = -1, y_0 = -1, \tag{2.13}$$

$$x_{-1} = -1, x_0 = A + 1, y_{-1} = -1, y_0 = A + 1, \tag{2.14}$$

$$x_{-1} = A + 1, x_0 = -1, y_{-1} = A + 1, y_0 = -1. \tag{2.15}$$

Proof: Since this proof is similar to the proof of Theorem 1, we therefore leave it to the readers.

Lemma 4. From (2.13)-(2.15), there is a three periodic cycle of system (1.1) as:

$$\{\cdots,(-1,-1),(-1,-1),(A+1,A+1),(-1,-1),\cdots\}.$$

Proof: Let $x_{-1}=-1, x_0=-1, y_{-1}=-1, y_0=-1$. Hence, we have from system (1.1):

$$x_1=x_{-1}y_0-1=(-1)\cdot(-1)+A=A+1,$$

$$y_1=y_{-1}x_0-1=(-1)\cdot(-1)+A=A+1,$$

$$x_2=x_0y_1-1=(-1)\cdot(A+1)+A=-1,$$

$$y_2=y_0x_1-1=(-1)\cdot(A+1)+A=-1,$$

$$x_3=x_1y_2-1=(A+1)\cdot(-1)+A=-1,$$

$$y_3=y_1x_2-1=(A+1)\cdot(-1)+A=-1,$$

$$x_4=x_2y_3-1=(-1)\cdot(-1)+A=A+1,$$

$$y_4=y_2x_3-1=(-1)\cdot(-1)+A=A+1.$$

Therefore, we procure the periodic cycle of system (1.1) with period three as:

$$\{(-1,-1),(-1,-1),(A+1,A+1),(-1,-1),\cdots\}.$$

So, the proof completed as desired.

In this here, we endeavour the boundedness of solutions of system (1.1). In particular, we investigate the behaviours of unbounded solutions of system (1.1). We also examine the invariant interval of solutions of system (1.1).

Lemma 5. Let $\{(x_n,y_n)\}_{n=-1}^{\infty}$ be a solution of system (1.1) and $A>-1$. Let $x_{-1}<-1, x_0<-1, y_{-1}<-1$ and $y_0<-1$. Then

$$x_1>A+1, y_1>A+1,$$

$$x_2<-1, y_2<-1,$$

$$x_3<-1, y_3<-1.$$

Proof: Assume that $x_{-1}<-1, x_0<-1, y_{-1}<-1$ and $y_0<-1$. Hence, we have from system (1.1):

$$x_1=x_{-1}y_0+A>A+1,$$

$$y_1=y_{-1}x_0+A>A+1,$$

$$x_2=x_0y_1+A<-1,$$

$$y_2 = y_0 x_1 + A < -1,$$

$$x_3 = x_1 y_2 + A < -1,$$

$$y_3 = y_1 x_2 + A < -1.$$

Note that, there is a significant role the following theorem for proof of Theorem 4.

Theorem 3. Let $\{(x_n, y_n)\}_{n=-1}^{\infty}$ be a solution of system (1.1). Then,

$$x_{n+3} - y_n = (x_{n+1} + 1)(y_{n+2} - y_n), \tag{2.16}$$

$$y_{n+3} - x_n = (y_{n+1} + 1)(x_{n+2} - x_n). \tag{2.17}$$

Proof: We get from system (1.1),

$$\begin{aligned} x_{n+3} - y_n &= (x_{n+1} y_{n+2} + A) - y_n \\ &= x_{n+1}(y_n x_{n+1} + A) + A - y_n \\ &= x_{n+1}^2 y_n + A x_{n+1} + A - y_n \\ &= y_n (x_{n+1}^2 - 1) + A(x_{n+1} + 1) \\ &= (x_{n+1} + 1)(y_n x_{n+1} - y_n + A) \\ &= (x_{n+1} + 1)(y_{n+2} - y_n). \end{aligned}$$

Similarly, (2.17) can also be easily proved. Thus, the proof completed as desired.

Theorem 4. Let $\{(x_n, y_n)\}_{n=-1}^{\infty}$ be a solution of system (1.1) and $A > -1$. Let $x_{-1} < -1, x_0 < -1, y_{-1} < -1$ and $y_0 < -1$. Then the following two statements hold, for $n = 0,1,2,\cdots,$:

i.

$$A + 1 < x_1 < y_4 < x_7 < \cdots < x_{6n+1} < y_{6n+4} < \cdots,$$

$$A + 1 < y_1 < x_4 < y_7 < \cdots < y_{6n+1} < x_{6n+4} < \cdots,$$

$$-1 > x_2 > y_5 > x_8 > \cdots > x_{6n+2} > y_{6n+5} > \cdots,$$

$$-1 > y_2 > x_5 > y_8 > \cdots > y_{6n+2} > x_{6n+5} > \cdots,$$

$$-1 > x_3 > y_6 > x_9 > \cdots > x_{6n+3} > y_{6n+6} > \cdots,$$

$$-1 > y_3 > x_6 > y_9 > \cdots > y_{6n+3} > x_{6n+6} > \cdots.$$

ii.

$$\lim_{n\to\infty} x_{6n+1} = \infty, \lim_{n\to\infty} y_{6n+1} = \infty,$$

$$\lim_{n\to\infty} x_{6n+2} = -\infty, \lim_{n\to\infty} y_{6n+2} = -\infty,$$

$$\lim_{n\to\infty} x_{6n+3} = -\infty, \lim_{n\to\infty} y_{6n+3} = -\infty,$$

$$\lim_{n\to\infty} x_{6n+4} = \infty,\ \lim_{n\to\infty} y_{6n+4} = \infty,$$

$$\lim_{n\to\infty} x_{6n+5} = -\infty,\ \lim_{n\to\infty} y_{6n+5} = -\infty,$$

$$\lim_{n\to\infty} x_{6n} = -\infty,\ \lim_{n\to\infty} y_{6n} = -\infty.$$

Proof: Let $x_{-1} < -1, x_0 < -1, y_{-1} < -1$ and $y_0 < -1$.

i. From Lemma 5, we know

$$x_1, y_1 > A + 1,$$

$$x_2, y_2 < -1,$$

$$x_3, y_3 < -1.$$

From Theorem 3 for $n = 1$, we have that,

$$x_4 - y_1 = (x_2 + 1)(y_3 - y_1).$$

From $x_2, y_3 < -1$ and $y_1 > 0$, we get

$$x_4 - y_1 > 0.$$

Hence, $x_4 > y_1 > 0$ and similarly $y_4 > x_1 > 0$. Thus, we obtain from Theorem 3 for $n = 2$,

$$x_5 - y_2 = (x_3 + 1)(y_4 - y_2).$$

From $y_2, x_3 < -1$ and $y_4 > 0$, we have that,

$$x_5 - y_2 < 0.$$

Thus, $x_5 < y_2 < -1$ and $y_5 < x_2 < -1$.
Therefore, we get from Theorem 3 for $n = 3$,

$$x_6 - y_3 = (x_4 + 1)(y_5 - y_3).$$

From system (1.1), we obtain that,

$$\begin{aligned} x_6 - y_3 &= (x_4 + 1)(y_5 - y_3) \\ &= (x_4 + 1)(y_3 x_4 + A - y_3) \\ &= (x_4 + 1)(y_3(x_2 y_3 + A) + A - y_3) \\ &= (x_4 + 1)(x_2 y_3^2 + (A - 1)y_3 + A). \end{aligned}$$

From $x_2 < -1$,

$$\begin{aligned} x_6 - y_3 &= (x_4 + 1)(x_2 y_3^2 + (A - 1)y_3 + A) \\ &< (x_4 + 1)(-y_3^2 + (A - 1)y_3 + A) \\ &= (x_4 + 1)(-y_3 + A)(y_3 + 1). \end{aligned}$$

Hence, from $A>-1, x_4>0$ and $y_3<-1$ we have

$$x_6-y_3<0.$$

Hence, $x_6<y_3<-1$ and likewise $y_6<x_3<-1$.
We have, from Theorem 3 for $n=4$,

$$x_7-y_4=(x_5+1)(y_6-y_4).$$

From $y_6, x_5<-1$ and $y_4>0$, we obtain,

$$x_7-y_4>0.$$

So, $x_7>y_4>x_1>0$ and similarly $y_7>x_4>y_1>0$.

Additionally, by induction, we obtain:

$$A+1<x_1<y_4<x_7<\cdots,$$

$$A+1<y_1<x_4<y_7<\cdots,$$

$$-1>x_2>y_5>x_8>\cdots,$$

$$-1>y_2>x_5>y_8>\cdots,$$

$$-1>x_3>y_6>x_9>\cdots,$$

$$-1>y_3>x_6>y_9>\cdots.$$

So, proof of (i) finished as desired.

ii. We have from system (1.1):

$$\begin{aligned} x_{6n+1} &= x_{6n-1}y_{6n}+A>x_{6n-1}y_{6n}-1 \\ &= (x_{6n-3}y_{6n-2}+A)(y_{6n-2}x_{6n-1}+A)-1 \\ &> (x_{6n-3}y_{6n-2}-1)(y_{6n-2}x_{6n-1}-1)-1 \\ &= x_{6n-3}y_{6n-2}^2x_{6n-1}-y_{6n-2}x_{6n-1}-x_{6n-3}y_{6n-2}. \end{aligned}$$

Thus, we get from $x_{6n-3}y_{6n-2}^2x_{6n-1}>0$ and $y_{6n-2}x_{6n-1}<0$,

$$x_{6n+1}>-x_{6n-3}y_{6n-2}.$$

Hence, we obtain from $x_{6n-3}<-1$,

$$\begin{aligned} x_{6n+1} &> y_{6n-2}=y_{6n-4}x_{6n-3}+A \\ &> y_{6n-4}x_{6n-3}-1 \\ &= y_{6n-4}(x_{6n-5}y_{6n-4}+A)-1 \\ &> y_{6n-4}(x_{6n-5}y_{6n-4}-1)-1 \\ &= x_{6n-5}y_{6n-4}^2-y_{6n-4}-1. \end{aligned}$$

Therefore, we have from $y_{6n-4} < -1$,

$$x_{6n+1} > x_{6n-5}. \tag{2.18}$$

So,

$$\lim_{n\to\infty} x_{6n+1} = \infty.$$

Similarly, it can be easily proved in other cases. Thus, we leave it to readers.
The following theorem shows that system (1.1) has an invariant interval.

Theorem 5. Assume all initial values of system (1.1) in $(-1,0)$ and $-1 < A < 0$. Then, $x_1 \in (-1,1)$ and $y_1 \in (-1,1)$ and for all $n \geq 2$, the solutions of system *(1.1)* are such that $x_n \in (-1,0)$ and $y_n \in (-1,0)$.

Proof: Let $x_{-1}, x_0, y_{-1}, y_0 \in (-1,0)$. We have from system (1.1):

$$x_1 = x_{-1}y_0 + A > x_{-1}y_0 - 1 > -1,$$

$$x_1 = x_{-1}y_0 + A < A + 1 < 1.$$

Hence, we obtain similarly $-1 < y_1 < 1$. Therefore, we get from $-1 < y_1 < 1$:

$$x_2 = x_0y_1 + A > -x_0 + A > -1,$$

$$\begin{aligned} x_2 = x_0y_1 + A &= x_0(y_{-1}x_0 + A) + A \\ &= y_{-1}x_0^2 + Ax_0 + A < A(x_0 + 1) < 0. \end{aligned}$$

Thus we have $-1 < x_2 < 0$ and $-1 < y_2 < 0$. Similarly calculations, we have $-1 < x_3 < 0$ and $-1 < y_3 < 0$. So, we have by induction, $x_n \in (-1,0)$ and $y_n \in (-1,0)$ for all $n \geq 2$.

Finally, we study the stability analysis of system (1.1). Moreover, we look for the stability of equilibrium point $(\bar{x}_1, \bar{y}_1)$ of system (1.1). Further, we examine the convergence behaviour of equilibrium point $(\bar{x}_1, \bar{y}_1)$ of related system. Additionally, in case of $A = \frac{1}{4}$, we investigate the stability of unique equilibrium point of system (1.1) . We also study the stability of equilibrium point $(\bar{x}_2, \bar{y}_2)$ of system (1.1).

Theorem 6. The following statements are true:

- **i.** If $-2 < A < \frac{1}{4}$, then the equilibrium point $(\bar{x}_1, \bar{y}_1)$ of system (1.1) is locally asymptotically stable.
- **ii.** If $A \leq -2$, then the equilibrium point $(\bar{x}_1, \bar{y}_1)$ of system (1.1) is locally unstable.

Proof: Firstly we consider the transformation for linearized form of system (1.1):

$$(x_n, x_{n-1}, y_n, y_{n-1}) \to (f, f_1, g, g_1),$$

where

$$f = x_{n-1}y_n + A,$$

$$f_1 = x_n,$$

$$g = y_{n-1}x_n + A,$$

$$g_1 = y_n.$$

Hence, the Jacobian matrix about equilibrium point $(\bar{x}_1, \bar{y}_1)$ is:

$$B(\bar{x},\bar{y}) = \begin{pmatrix} 0 & \bar{y}_1 & \bar{x}_1 & 0 \\ 1 & 0 & 0 & 0 \\ \bar{y}_1 & 0 & 0 & \bar{x}_1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Therefore, the linearized system about the equilibrium point $(\bar{x}_1, \bar{y}_1)$ is $X_{N+1} = B(\bar{x},\bar{y})X_n$ where $X_n = \left((x_n, x_{n-1}, y_n, y_{n-1})\right)^T$ and

$$B(\bar{x},\bar{y}) = \begin{pmatrix} 0 & \frac{1-\sqrt{1-4A}}{2} & \frac{1-\sqrt{1-4A}}{2} & 0 \\ 1 & 0 & 0 & 0 \\ \frac{1-\sqrt{1-4A}}{2} & 0 & 0 & \frac{1-\sqrt{1-4A}}{2} \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

So, the characteristic equation of $B(\bar{x},\bar{y})$ is

$$f(\lambda) = \lambda^4 + \left(A - \frac{3}{2} + \frac{3}{2}\sqrt{1-4A}\right)\lambda^2 + \left(-A + \frac{1}{2} - \frac{1}{2}\sqrt{1-4A}\right) = 0. \tag{2.19}$$

Therefore, the roots of characteristic equation (2.19) are

$$\lambda_1 = \frac{1}{2}\sqrt{3 - 2A - 3\sqrt{1-4A} + \sqrt{4A^2 + 12A\sqrt{1-4A} - 32A - 10\sqrt{1-4A} + 10}},$$

$$\lambda_2 = -\frac{1}{2}\sqrt{3 - 2A - 3\sqrt{1-4A} + \sqrt{4A^2 + 12A\sqrt{1-4A} - 32A - 10\sqrt{1-4A} + 10}},$$

$$\lambda_3 = \frac{1}{2}\sqrt{3 - 2A - 3\sqrt{1-4A} - \sqrt{4A^2 + 12A\sqrt{1-4A} - 32A - 10\sqrt{1-4A} + 10}},$$

$$\lambda_4 = -\frac{1}{2}\sqrt{3 - 2A - 3\sqrt{1-4A} - \sqrt{4A^2 + 12A\sqrt{1-4A} - 32A - 10\sqrt{1-4A} + 10}}.$$

i. From λ_1, we have

$$\lambda_1 = \frac{1}{2}\sqrt{3 - 2A - 3\sqrt{1-4A} + \sqrt{4A^2 + 12A\sqrt{1-4A} - 32A - 10\sqrt{1-4A} + 10}}$$

$$= \frac{1}{2}\sqrt{3 - 2A - 3\sqrt{1-4A} + \sqrt{\left(1 + 2A + 3\sqrt{1-4A}\right)^2 - 16\sqrt{1-4A}}}.$$

From $-2 < A < \frac{1}{4}$, we obtain that

$$\lambda_1 < \frac{1}{2}\sqrt{3 - 2A - 3\sqrt{1-4A} + \sqrt{\left(1 + 2A + 3\sqrt{1-4A}\right)^2}}.$$

Hence,

$$\begin{aligned}\lambda_1 &< \frac{1}{2}\sqrt{3 - 2A - 3\sqrt{1-4A} + \left|1 + 2A + 3\sqrt{1-4A}\right|}\\ &= \frac{1}{2}\sqrt{3 - 2A - 3\sqrt{1-4A} + 1 + 2A + 3\sqrt{1-4A}}\\ &= \frac{1}{2}\sqrt{4} = 1.\end{aligned}$$

So, we get that $0 < \lambda_1 < 1$. Moreover, since $\lambda_3 < \lambda_1$, we have $0 < \lambda_3 < \lambda_1 < 1$. Similarly the other roots lie inside unit disk too. Thus, all roots of characteristic equation (2.19) lie inside the open unit disk $|\lambda| < 1$. So, if $-2 < A < \frac{1}{4}$ then the equilibrium point $(\bar{x}_1, \bar{y}_1)$ of system (1.1) is locally asymptotically stable.

ii. In this here, we consider the characteristic equation (2.19) such that

$$f(\lambda) = \lambda^4 + \left(A - \frac{3}{2} + \frac{3}{2}\sqrt{1-4A}\right)\lambda^2 + \left(-A + \frac{1}{2} - \frac{1}{2}\sqrt{1-4A}\right)$$

Thus, we know that $f(\lambda)$ has four roots and so absolute value of the product of these roots is

$$|\lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot \lambda_4| = \left|-A + \frac{1}{2} - \frac{1}{2}\sqrt{1-4A}\right| = \frac{\left(1 - \sqrt{1-4A}\right)^2}{4} \geq 1.$$

Hence, from $A \leq -2$, we obtain that at least one of $|\lambda_1|, |\lambda_2|, |\lambda_3|$ and $|\lambda_4|$ is greater than one. Consequently, if $A \leq -2$ then the equilibrium point $(\bar{x}_1, \bar{y}_1)$ of system (1.1) is locally unstable.

The next theorem shows that if all initial values are in $(-1,0)$ and $-1 < A < 0$ then every solutions of system (1.1) tend to equilibrium point $(\bar{x}_1, \bar{y}_1)$.

Theorem 7. Let $x_{-1}, x_0, y_{-1}, y_0 \in (-1,0)$ and $-1 < A < 0$. Then every solutions of system (1.1) converge to equilibrium point $(\bar{x}_1, \bar{y}_1)$ of system (1.1).

Proof: Assume that $x_{-1}, x_0, y_{-1}, y_0 \in (-1,0)$. Thus, every solutions of system (1.1) is bounded according to Theorem 5. According to this, $x_n \in (-1,0)$ and $y_n \in (-1,0)$ for $n \geq 2$. Hence, we get

$$\begin{aligned} L_1 &= \limsup_{n\to\infty} x_n, \quad L_2 = \limsup_{n\to\infty} y_n, \\ l_1 &= \liminf_{n\to\infty} x_n, \quad l_2 = \liminf_{n\to\infty} y_n. \end{aligned} \tag{2.20}$$

Hence, we have from system (1.1) and (2.20):

$$L_1 \le L_1 \cdot l_2 + A, \tag{2.21}$$

$$l_1 \ge l_1 \cdot L_2 + A, \tag{2.22}$$

$$L_2 \le L_2 \cdot l_1 + A, \tag{2.23}$$

$$l_2 \ge l_2 \cdot L_1 + A. \tag{2.24}$$

Thus, we obtain from (2.21) and (2.24)

$$-A + l_2 \le L_1 \cdot l_2 \le -A + L_1,$$

hence

$$L_1 \le l_2. \tag{2.25}$$

Additionally, we get from (2.22) and (2.23)

$$L_2 \le l_1. \tag{2.26}$$

Moreover, we know,

$$l_1 \le L_1 \quad \text{and} \quad l_2 \le L_2. \tag{2.27}$$

So, we have from (2.25), (2.26) and (2.27)

$$l_1 \le L_1 \le l_2 \le L_2 \le l_1.$$

Hence, we obtain the following equalities:

$$l_1 = L_1 = l_2 = L_2.$$

So, we have the followings:

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = l.$$

Therefore, we have from system (1.1):

$$l = \bar{x}_1 = \bar{y}_1,$$

$$l = \bar{x}_2 = \bar{y}_2.$$

Due to $\bar{x}_2, \bar{y}_2 \notin (-1,0)$, we have $l = \bar{x}_1 = \bar{y}_1$. Thus, if the initial values in $(-1,0)$ then every solution of system (1.1) tends to negative equilibrium point $(\bar{x}_1, \bar{y}_1)$. Therefore, the proof completed.

Theorem 8. Let $A = \frac{1}{4}$. The unique equilibrium point $(\bar{x}, \bar{y}) = \left(\frac{1}{2}, \frac{1}{2}\right)$ of system (1.1) is locally unstable. Because it is nonhyperbolic equilibrium point.

Proof: First of all, if $A = \frac{1}{4}$ then system (1.1) turns into the following system:

$$x_{n+1} = x_{n-1}y_n + \frac{1}{4}, y_{n+1} = y_{n-1}x_n + \frac{1}{4}, n = 0,1, \ldots. \tag{2.28}$$

Now, we deal with linearized form of system (2.28). Because of this, we take the transformation:

$$(x_n, x_{n-1}, y_n, y_{n-1}) \to (f, f_1, g, g_1),$$

where

$$f = x_{n-1}y_n + \frac{1}{4},$$

$$f_1 = x_n,$$

$$g = y_{n-1}x_n + \frac{1}{4},$$

$$g_1 = y_n.$$

Thus, we get the Jacobian matrix about equilibrium point $(\bar{x}, \bar{y}) = \left(\frac{1}{2}, \frac{1}{2}\right)$:

$$B(\bar{x}, \bar{y}) = \begin{pmatrix} 0 & \bar{y} & \bar{x} & 0 \\ 1 & 0 & 0 & 0 \\ \bar{y} & 0 & 0 & \bar{x} \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Hence, the linearized system about the equilibrium point $(\bar{x}, \bar{y}) = \left(\frac{1}{2}, \frac{1}{2}\right)$ is $X_{N+1} = B(\bar{x}, \bar{y})X_n$ where

$$X_n = \big((x_n, x_{n-1}, y_n, y_{n-1})\big)^T$$

and

$$B(\bar{x}, \bar{y}) = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Therefore, the characteristic equation of $B(\bar{x}, \bar{y})$ is

$$\lambda^4 - \frac{5}{4}\lambda^2 + \frac{1}{4} = 0 \tag{2.29}$$

So, we obtain four roots of Eq.(2.29):

$$\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}, \lambda_3 = -1, \lambda_4 = 1.$$

Hence,

$$|\lambda_{1,2}| < 1 = |\lambda_{3,4}|.$$

Owing to linearized stability theorem, the unique equilibrium point $(\bar{x}, \bar{y})$ of system (1.1) is locally unstable.

Theorem 9. Equilibrium point $(\bar{x}_2, \bar{y}_2)$ of system (1.1) is locally unstable for $A < \frac{1}{4}$.

Proof: Now, we take the transformation $(x_n, x_{n-1}, y_n, y_{n-1}) \to (f, f_1, g, g_1)$ such that

$$f = x_{n-1}y_n + A,$$

$$f_1 = x_n,$$

$$g = y_{n-1}x_n + A,$$

$$g_1 = y_n.$$

Thus, we obtain the Jacobian matrix about equilibrium point $(\bar{x}_2, \bar{y}_2)$:

$$B(\bar{x}, \bar{y}) = \begin{pmatrix} 0 & \bar{y}_2 & \bar{x}_2 & 0 \\ 1 & 0 & 0 & 0 \\ \bar{y}_2 & 0 & 0 & \bar{x}_2 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Therefore, the linearized system about the equilibrium point $(\bar{x}_2, \bar{y}_2)$ is $X_{N+1} = B(\bar{x}, \bar{y})X_n$ where $X_n = \left((x_n, x_{n-1}, y_n, y_{n-1})\right)^T$ and

$$B(\bar{x}, \bar{y}) = \begin{pmatrix} 0 & \frac{1+\sqrt{1-4A}}{2} & \frac{1+\sqrt{1-4A}}{2} & 0 \\ 1 & 0 & 0 & 0 \\ \frac{1+\sqrt{1-4A}}{2} & 0 & 0 & \frac{1+\sqrt{1-4A}}{2} \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Hence, we obtain the characteristic equation of $B(\bar{x}, \bar{y})$ as

$$\lambda^4 + \left(A - \frac{3}{2} - \frac{3}{2}\sqrt{1-4A}\right)\lambda^2 - A + \frac{1}{2} + \frac{1}{2}\sqrt{1-4A} = 0. \quad (2.30)$$

Now we take

$$P(x) = x^4 + \left(A - \frac{3}{2} - \frac{3}{2}\sqrt{1-4A}\right)x^2 - A + \frac{1}{2} + \frac{1}{2}\sqrt{1-4A}.$$

Therefore,

$$P(1) = P(-1) = -\sqrt{1-4A} < 0,$$

$$P(0) = -A + \frac{1}{2} + \frac{1}{2}\sqrt{1-4A}$$
$$= \frac{1}{4}\left(\sqrt{1-4A}+1\right)^2 > 0$$

and

$$\lim_{x\to\infty} P(x) = +\infty,$$

$$\lim_{x\to-\infty} P(x) = +\infty.$$

From these we have four roots of $P(x)$ such that

$$-\infty < x_1 < -1 < x_2 < 0 < x_3 < 1 < x_4 < \infty.$$

Therefore, the absolute values of two roots of $P(x)$ are greater than one. From this and from linearized stability theorem, the equilibrium points $(\bar{x}_2, \bar{y}_2)$ of system (1.1) is locally unstable.

3. NUMERICAL EXAMPLES

Up to this section, we proved some theoretical results for system (1.1). Now, we give some numerical examples in order to verify our results.

Example 1 Let $A = -\frac{2}{3}$ and the initial values $x_{-1} = \bar{x}_1, x_0 = \bar{x}_2, y_{-1} = \bar{y}_2, y_0 = \bar{y}_1$. If $A = -\frac{2}{3}$ then $\bar{x}_1 = \bar{y}_1 = \frac{1-\sqrt{\frac{11}{3}}}{2} \approx -0.46$ and $\bar{x}_2 = \bar{y}_2 = \frac{1+\sqrt{\frac{11}{3}}}{2} \approx 1.46$. Thus, there are two periodic solutions of the system (1.1) with prime period. The Fig. 1 verifies results of Theorem 1 and two periodic cycle of system is

$$\{\cdots, (\bar{x}_1, \bar{y}_2), (\bar{x}_2, \bar{y}_1), (\bar{x}_1, \bar{y}_2), \cdots\}.$$

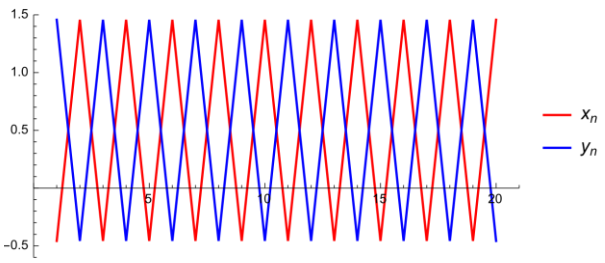
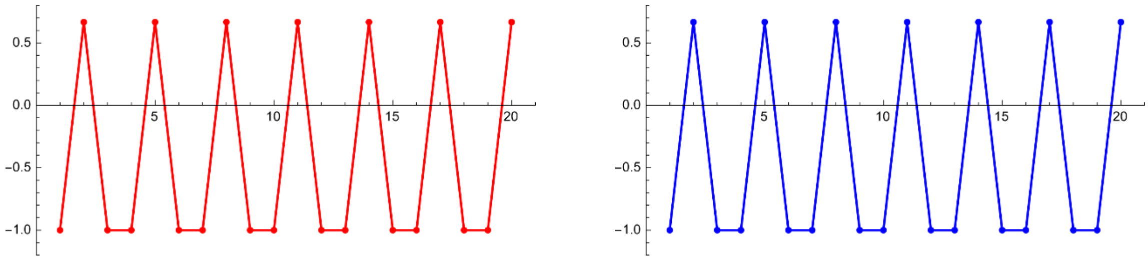


Figure 1. Plot of system (1.1) for initial values $x_{-1} = \bar{x}_1$, $x_0 = \bar{x}_2$, $y_{-1} = \bar{y}_2$, $y_0 = \bar{y}_1$.

Example 2. Let $A = -\frac{1}{3}$ and the initial values $x_{-1} = A+1, x_0 = -1, y_{-1} = A+1, y_0 = -1$. Thus, there are three periodic solutions of the system (1.1) with prime period. The Fig. 2 (a and b) verify results of Theorem 2 and three periodic cycle of system is

$$\{\cdots,(A+1,A+1),(-1,-1),(-1,-1),\cdots\}.$$



a) Plot of x_n **b) Plot of y_n**

Figure 2. Plot of system (1.1) for initial values $x_{-1}=A+1$, $x_0=-1$, $y_{-1}=A+1$, $y_0=-1$.

Example 3. Let $A=-\frac{1}{2}$ and the initial values $x_{-1}=-0.9, x_0=-0.2, y_{-1}=-0.1, y_0=-0.8$. Thus, solution of system (1.1) converges to negative equilibrium point $(\bar{x}_1,\bar{y}_1)=\left(\frac{1-\sqrt{3}}{2},\frac{1-\sqrt{3}}{2}\right)$. The Fig. 3 verifies to results of Theorem 7.

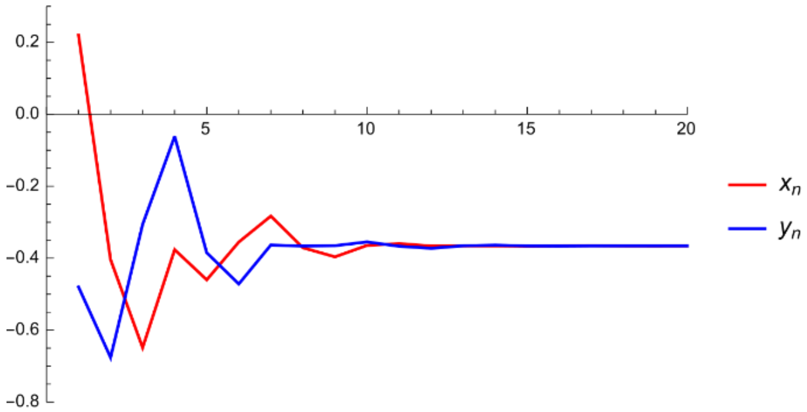


Figure 3. Plot of system (1.1).

4. CONCLUSION

This paper investigate the dynamics of system (1.1). Especially we find out the periodicity of system (1.1) with period both two and three. We also study the stability of equilibrium points of system (1.1). We discover that if $-2<A<\frac{1}{4}$, then the equilibrium point $(\bar{x}_1,\bar{y}_1)$ of system (1.1) is locally asymptotically stable otherwise if $A\leq -2$, then the equilibrium point $(\bar{x}_1,\bar{y}_1)$ of system (1.1) is locally unstable. Additionally, if the initial values in $(-1,0)$ and $-1<A<0$, then every solutions of system (1.1) converge to equilibrium point $(\bar{x}_1,\bar{y}_1)$ of related system. Moreover, we show that both the equilibrium point $(\bar{x},\bar{y})=\left(\frac{1}{2},\frac{1}{2}\right)$ and equilibrium point $(\bar{x}_2,\bar{y}_2)$ of system (1.1) is locally unstable.

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